

EXPLICIT AMBIENT METRICS AND HOLONOMY

IAN M. ANDERSON, THOMAS LEISTNER, AND PAWEŁ NUROWSKI

ABSTRACT. We present three large classes of examples of conformal structures for which the equations for the Fefferman-Graham ambient metric to be Ricci-flat are linear PDEs, which we solve explicitly. These explicit solutions enable us to discuss the holonomy of the corresponding ambient metrics. Our examples include conformal pp-waves and, more importantly, conformal structures that are defined by generic rank 2 and 3 distributions in respective dimensions 5 and 6. The corresponding explicit Fefferman-Graham ambient metrics provide a large class of metrics with holonomy equal to the exceptional non-compact Lie group \mathbf{G}_2 as well as ambient metrics with holonomy contained in $\mathbf{Spin}(4, 3)$.

1. INTRODUCTION

Let g_0 be a smooth semi-Riemannian metric on an n -dimensional manifold M , with $n \geq 2$ and with local coordinates (x^i) . A *pre-ambient metric* for g_0 is, by definition, a smooth semi-Riemannian metric \tilde{g} of the form

$$(1.1) \quad \tilde{g} = 2 \, dt d(\varrho t) + t^2 g(x^i, \varrho)$$

defined on the *ambient space*

$$\tilde{M} = (0, +\infty) \times M \times (-\epsilon, \epsilon), \quad \epsilon > 0,$$

with coordinates (t, x^i, ϱ) and such that $g(x^i, \varrho)|_{\varrho=0} = g_0(x^i)$. We call a pre-ambient metric an *ambient metric* if

$$(1.2) \quad Ric(\tilde{g}) = 0.$$

It is a fundamental observation of Fefferman and Graham [8, 9] that conformally equivalent metrics have the same ambient metrics. More precisely, given a metric g_0 with ambient metric \tilde{g} and a conformally equivalent metric \hat{g}_0 in the conformal class $[g_0] = \{e^{2\phi} g_0 \mid \phi \in C^\infty(M)\}$ of g_0 , an ambient metric for \hat{g}_0 can be obtained by pulling back \tilde{g} by a diffeomorphism of \tilde{M} . The importance of the ambient metric is that the semi-Riemannian invariants of \tilde{g} give rise to conformal invariants of g_0 .

Given a pre-ambient metric as in (1.1) we call the system of PDEs for an unknown ϱ -dependent family of metrics $g(x^i, \varrho)$ the *Fefferman-Graham equations*. Their explicit form is given in [9, eq. 3.17]. In general we assume ambient metrics to be smooth, however, as we attempt in our examples to find the most general solutions to the Fefferman-Graham equations, we will also present Ricci-flat metrics of the form (1.1) that are defined only for $\varrho \geq 0$ and differentiable only to finite order at $\varrho = 0$. By a slight abuse of terminology, we will call them *non-smooth ambient metrics*.

2010 *Mathematics Subject Classification.* Primary: 53C29, 53A30; secondary: 53C50.

Key words and phrases. Conformal structures, Fefferman-Graham ambient metric, exceptional holonomy, generic distributions, pp-waves.

Support for this research was provided by National Science Grant ACI 1148331SI2-SSE (IA), by the Australian Research Council via the grants FT110100429 and DP120104582, and by the Polish National Science Center (NCN) via DEC-2013/09/B/ST1/01799.

Regarding the existence and uniqueness of ambient metrics, Fefferman and Graham [8, 9] have shown the following: If g_0 is a smooth metric in odd dimension n , then there is always a smooth pre-ambient metric with

$$(1.3) \quad \tilde{\nabla}^k Ric(\tilde{g})|_{\varrho=0} = 0,$$

for all $k = 0, 1, \dots$, and in which $\tilde{\nabla}$ is the Levi-Civita connection of \tilde{g} . The proof of this result proceeds by constructing $g(x^i, \varrho)$ from the initial conditions as a power series in ϱ . The coefficients of this power series are determined by equation (1.3) and thus yield a formal solution to the Fefferman-Graham equations (1.2). One can show that the obtained formal power series is convergent whenever g_0 is analytic, and hence, in this case one gets an ambient metric as defined above. For analytic g_0 this is the *unique analytic* ambient metric.

When $n > 2$ is even, the situation is more subtle, and in general one can only find pre-ambient metrics satisfying the condition (1.3) for $k = 0, \dots, \frac{n}{2} - 1$. For (1.3) to be satisfied beyond the order $k = \frac{n}{2} - 1$ one checks if a certain symmetric $(2, 0)$ -tensor, the so-called *Fefferman-Graham obstruction tensor* $\mathcal{O}_{ij}(g_0)$, vanishes. Fefferman and Graham showed that the vanishing of $\mathcal{O}_{ij}(g_0)$ is equivalent to the existence of a pre-ambient metric satisfying condition (1.3) up to all orders k , and hence for analytic g_0 's, equivalent to the existence of an ambient metric in our sense. However, when n is even and $\mathcal{O}_{ij}(g_0)$ vanishes, analytic ambient metrics are not unique.

Unfortunately, for general metrics g_0 the Fefferman-Graham equations (1.2) are a complicated non-linear system of partial differential equations for $g(x^i, \varrho)$ which in general cannot be solved explicitly. This makes it very hard to find explicit ambient metrics. Indeed, prior to the results of [15, 16], the only known explicit examples of ambient metrics were related to Einstein metrics g_0 or certain products of Einstein metrics [14, 18, 10]. In fact, if g_0 is an Einstein metric, i.e., $Ric(g_0) = \Lambda g_0$, then an ambient metric is given by

$$(1.4) \quad \tilde{g} = 2dtd(\varrho t) + t^2 \left(1 + \frac{\Lambda \varrho}{2(n-1)} \right)^2 g_0.$$

In other words, for an Einstein metric g_0 , a solution to the Fefferman-Graham equations is given by $g(x^i, \varrho) = (1 + \frac{\Lambda}{2(n-1)} \varrho)^2 g_0$.

The goal of this paper is to present some interesting classes of examples of metrics g_0 for which, quite remarkably, *the Fefferman-Graham equations reduce to linear PDEs*. The more general class of metrics with linear Fefferman-Graham equations will be described in our subsequent article [1]. Our approach for solving the Fefferman-Graham equations in our examples, either in closed form or by standard power series methods, is based on a particular feature of the equations. The equations are of second order, but, since they have a singular point at $\varrho = 0$, the first ϱ -derivative of $g(x, \varrho)$ at $\varrho = 0$ cannot be chosen freely. Instead, it is determined as $g'|_{\varrho=0} = 2P$, where P is the *Schouten tensor*

$$P = \frac{1}{n-2} \left(Ric(g_0) - \frac{Scal(g_0)}{2(n-1)} g_0 \right)$$

of g_0 . The common feature of the considered examples is that the Schouten tensor of g_0 is nilpotent (when considered as $(1, 1)$ -tensor), and we postulate the following ansatz for $g(\varrho, x^i)$

$$g(x^i, \varrho) = g_0 + h(x^i, \varrho),$$

where $h(x^i, \varrho)$ is required to have the *same image and kernel as P* for all ϱ (again considered as $(1, 1)$ -tensors). Then the Fefferman-Graham equations turn out to be *linear* second order PDEs for the unknown symmetric tensor $h = h(x^i, \varrho)$.

The conformal structures presented here are of three different types. The first class of examples, which are really a *prototype* for all the conformal structures for which the Fefferman-Graham equations become linear, are the so-called *conformal pp-waves*. These are presented in Section 2.

We extend our results from [15] to other signatures and by determining all solutions to (1.2). Hence, we find ambient metrics which, when n is odd, are defined for $\varrho \geq 0$ and only $\frac{n}{2} - 1$ -times differentiable in ϱ at $\varrho = 0$. When n is even, the obtained ambient metrics are only defined on the domain $\varrho > 0$ as they contain logarithmic terms. An upper bound for the holonomy algebras of the ambient metrics is determined.

The second class of examples, discussed in Section 3, arises from conformal structures in signature $(3, 2)$ defined by generic rank 2 distributions \mathcal{D} in dimension 5 [20]. These distributions can be brought into normal form

$$(1.5) \quad \mathcal{D} = \text{Span}\left(\partial_q, \partial_x + p \partial_y + q \partial_p + F \partial_z\right),$$

where our coordinates on \mathbb{R}^5 are (x, y, z, p, q) , we write $\partial_q = \frac{\partial}{\partial q}$ etc., and $F = F(x, y, z, p, q)$ is a smooth function with $\partial_q^2(F) \neq 0$. The most remarkable aspect of the associated conformal structures is that their normal conformal Cartan connection reduces to the exceptional 14-dimensional, simple, non-compact Lie group \mathbf{G}_2 (as shown in [20]) and hence have their conformal holonomy¹ contained in this group. In [16] we showed that for distributions defined by $F = q^2 + \sum_{i=0}^6 a_i p^i + bz$, with constants a_i and b , also the holonomy of the associated analytic ambient metric is contained in, and generically equal to, \mathbf{G}_2 . The results in [16] were subsequently generalized in [12] to *all* conformal structures defined by analytic distributions \mathcal{D} as in (1.5), by showing that their unique analytic ambient metric admits a certain parallel 3-form (or equivalently, a parallel non-null spinor field) whose stabilizer is \mathbf{G}_2 . This result shows that not only the conformal holonomy but also the holonomy of the ambient metric is contained in \mathbf{G}_2 . Also in [12] a sufficient criterion was given for the holonomy of the ambient metric being equal to \mathbf{G}_2 , however, the examples in [16] with holonomy equal to \mathbf{G}_2 do *not* satisfy this criterion, and in general the criterion is very difficult to check. Here we generalize the results of [16] to a much larger class. In fact, for all conformal classes associated to distributions \mathcal{D} given by a function F of the following form,

$$(1.6) \quad F = q^2 + f(x, y, p) + h(x, y)z,$$

the Fefferman-Graham equations turn out to be linear second order PDEs. For such conformal classes, in Section 3.2 we will show the following:

- (i) We provide the most general formal solutions (in terms of power series) to the Fefferman-Graham equations, including those that are only twice differentiable in ϱ at $\varrho = 0$.
- (ii) We derive a linear *first order* system that has the same (analytic) solutions as the Fefferman-Graham equations.
- (iii) Using this first order system, we derive explicit formulae for the 3-form and the parallel spinor field for the analytic ambient metric.
- (iv) For particular choices of f and h we give explicit (closed form) ambient metrics.
- (v) We give an explicit, easy to check criterion that ensures that the ambient metric has holonomy *equal* to \mathbf{G}_2 .

The first order system in (ii) is crucial as it gives us a tool, leaving the conformal context aside, to construct metrics in dimension 7 with holonomy contained in \mathbf{G}_2 . Together with the criterion in (v) it enables us to construct a large class of metrics with holonomy *equal* to \mathbf{G}_2 and depending on the functions $f = f(x, y, p)$ and $h = h(x, y)$. The examples we obtain include, but are not restricted to, ambient metrics which are polynomial functions in the ambient space variable ϱ of

¹We point out that all of our considerations are *local* in the sense that we work on simply connected manifolds. This implies that the holonomy groups we encounter are connected and hence can be identified with their Lie algebras. Moreover, throughout the paper \mathfrak{g}_2 and \mathbf{G}_2 refer to the *split real form* of the exceptional simple Lie algebra of type G_2 and to the corresponding connected Lie group in $\mathbf{SO}(4, 3)$.

any prescribed order². Furthermore, we discuss the holonomy of the non-smooth solutions and surprisingly find metrics with holonomy equal to \mathbf{G}_2 amongst them.

The third class of examples, which are altogether new and presented in Section 4, are derived from Bryant's conformal structures [4]. They are naturally associated with generic rank 3-distributions in dimension 6, i.e., rank 3 distributions \mathcal{D} on a 6-manifold M with $[\mathcal{D}, \mathcal{D}] + \mathcal{D} = TM$. According to [4], to such distributions one can associate a conformal structure in dimension 6, whose *conformal holonomy* is now contained in $\mathbf{Spin}(4, 3)$. In this paper we consider a certain family of such distributions locally on \mathbb{R}^6 , which is given in terms of a differentiable function $f = f(x^1, x^2, x^3)$ of *only three variables* (x^1, x^2, x^3) satisfying genericity condition $\frac{\partial f}{\partial x^3} \neq 0$. Explicitly, the distributions from the family are given by

$$(1.7) \quad \mathcal{D} = \text{Span} \left(\frac{\partial}{\partial x^3} - x^2 \frac{\partial}{\partial y^1}, \frac{\partial}{\partial x^1} - f(x^1, x^2, x^3) \frac{\partial}{\partial y^2}, \frac{\partial}{\partial x^2} - x^1 \frac{\partial}{\partial y^3} \right),$$

where $(x^1, x^2, x^3, y^1, y^2, y^3)$ are coordinates on \mathbb{R}^6 . We show that for conformal structures defined by \mathcal{D} with $f = f(x^1, x^3)$ or $f = f(x^2, x^3)$ the Fefferman-Graham obstruction tensor $\mathcal{O}_{ij}(g_0)$ vanishes. However, generically they are not conformally Einstein. With the vanishing of the obstruction tensor, we are assured that for the conformal classes associated with such \mathcal{D} 's, the Fefferman-Graham equations have an analytic solution, whenever f is analytic. Again the Fefferman-Graham equations turn out to be *linear*, which enables us to find the ambient metrics explicitly. To our knowledge, these are the first examples of ambient metrics for Bryant's conformal classes. However, in this case, as the general theory for even dimensions predicts, they are not unique anymore. Now, new Ricci flat ambient metrics are given and parametrized by a symmetric, trace-free $(2, 0)$ -tensor $\varrho^3 Q$.

For the ambient metrics of Bryant's conformal classes defined by $f = f(x^1, x^3)$ or $f = f(x^2, x^3)$ we give an upper bound for the holonomy of the ambient metric by showing that it has to be contained in Poincaré group in signature $(3, 3)$,

$$\mathbf{PO}(3, 3) = \mathbf{SO}(3, 3) \ltimes \mathbb{R}^{3, 3},$$

which is the stabilizer of a decomposable 4-form defining an invariant totally null 4-plane. This implies that these ambient metrics cannot have holonomy *equal to* $\mathbf{Spin}(4, 3)$. Furthermore, we show that for ambient metrics with $Q = 0$ the holonomy reduces further to

$$\mathbf{PO}(3, 2) = \mathbf{SO}(3, 2) \ltimes \mathbb{R}^{3, 2},$$

which in turn is *contained in* $\mathbf{Spin}(4, 3)$. Note that the representations of the Poincaré groups in $\mathbf{SO}(4, 4)$ are *not* the usual representations as stabilizers of null vectors. Moreover we compute three examples of ambient metrics, including one with $Q \neq 0$. In the example with $Q \neq 0$, the ambient holonomy is equal to $\mathbf{PO}(3, 3)$, and hence, is *not contained in* $\mathbf{Spin}(4, 3)$. The other two examples of ambient metrics have $Q = 0$, one with holonomy equal to $\mathbf{PO}(3, 2)$, and the other with the 7-dimensional Heisenberg group as holonomy. We believe that these ambient metrics reveal some interesting 4-form geometry in signature $(4, 4)$, which suggests further investigation.

Finally, we give an explicit example of a conformal structure, associated with a rank 3-distribution as in (1.7) having $f = x^3 + x^1 x^2 + (x^2)^2 + (x^3)^2$, for which the obstruction tensor does *not* vanish. This example shows that the vanishing of the obstruction tensor \mathcal{O} for the conformal classes associated with $f = f(x^1, x^3)$ or $f = f(x^2, x^3)$ is exceptional and is associated with the chosen symmetry, rather than the general feature of distributions with f depending on the three variables (x^1, x^2, x^3) . Again, many open questions remain, for example: What are the conditions on f that

²We should mention that during the preparation of this paper, the article [22] appeared in which it was shown that the conformal class arising from the distribution found by Doubrov and Govorov [7] with $F = q^{1/3} + y$, defined on a domain with $q > 0$, has a (non-polynomial) ambient metric with holonomy equal to \mathbf{G}_2 .

are equivalent to the vanishing of the obstructions tensor, and in case of vanishing obstruction, how many ambient metrics are there with holonomy (contained in) $\mathbf{Spin}(4, 3)$?

Many of the calculations for this paper were performed using the Maple *DifferentialGeometry* package. Sample worksheets, illustrating the results of this paper, can be found at http://digitalcommons.usu.edu/dg_applications/.

Acknowledgements. We thank Travis Willse for comments on the first version of this article.

2. GENERALIZED CONFORMAL PP-WAVES AND THEIR AMBIENT METRICS

2.1. Generalized pp-waves. In this section we provide explicit formulae for ambient metrics for conformal structures on pseudo-Riemannian manifolds which, in Lorentzian signature, are known as *pp-waves*. The formulae presented here generalize our results on conformal pp-waves in [15]. Lorentzian pp-wave metrics are defined on an open set $\mathcal{U} \subset \mathbb{R}^n$, with coordinates (x^i, u, v) , $i = 1, 2, \dots, n-2$, where they assume the form

$$g_H = 2du(dv + H(x^i, u) du) + \sum_{i=1}^{n-2} (dx^i)^2.$$

Here $H = H(x^i, u)$ is a real function on \mathcal{U} which specifies a pp-wave. For Lorentzian pp-waves of odd dimension we have determined the unique ambient metric that is analytic in ϱ in [15] as

$$\tilde{g}_H = 2d(t\varrho) dt + t^2 \left(g + \left(\sum_{k=1}^{\infty} \frac{\Delta^k H}{k! p_k} \varrho^k \right) du^2 \right),$$

where $p_k = \prod_{j=1}^k (2j - n)$ and $\Delta = \sum_{i=1}^{n-2} \partial_i^2$ is the flat Laplacian. When n is even, this formula also gives analytic ambient metrics if one assumes that $\Delta^{\frac{n}{2}} H = 0$. Here we generalise this result to higher signature and, more importantly, determine all solutions to the Fefferman-Graham equations.

Let $1 \leq p \leq n/2$ be a natural number and let the indices run as follows: $A, B, C \dots \in \{1, \dots, n-2p\}$ $a, b, c \dots \in \{1, \dots, p\}$. Let

$$\mathcal{U} \subset \mathbb{R}^n \ni (x^1, \dots, x^{n-2p}, u^1, \dots, u^p, v^1, \dots, v^p)$$

be an open set, and $H_{ac} = H_{ca}$ smooth functions on \mathcal{U} satisfying and $\frac{\partial}{\partial v^b}(H_{ac}) = 0$. Then we define a *generalized pp-wave* in signature $(p, n-p)$ (for short *pp-wave*), as follows:

$$(2.1) \quad g_H = 2du^a(\delta_{ab}dv^b + H_{ab}du^b) + \delta_{AB}dx^A dx^B,$$

where we denote $H = (H_{ac})_{a,c=1}^p$. Such pp-waves admit p parallel null vector fields $\frac{\partial}{\partial v^a}$. The scalar curvature of pp-waves vanishes and their Ricci tensor is given as

$$Ric(g_H) = -\Delta(H_{ac})du^a du^c,$$

with $\Delta = \delta^{AB}\partial_A\partial_B$ the flat Laplacian with respect to the x^A -coordinates. Hence, the Schouten tensor of a pp-wave is given by

$$(2.2) \quad P = -\frac{1}{n-2} \Delta(H_{ac}) du^a du^c,$$

and therefore is 2-step nilpotent (i.e. $P^2 = 0$), when considered as a $(1, 1)$ -tensor.

2.2. Analytic and non-analytic ambient metrics for pp-waves. The special form of the Schouten tensor in (2.2) of a metric g_H in (2.1) suggests the following ansatz for $g(x^A, u^b, v^c, \varrho)$ (which we will motivate further in [1]) for the ambient metric of the conformal class $[g_H]$:

$$(2.3) \quad \tilde{g}_{H,h} = 2d(\varrho t)dt + t^2(g_H + 2h_{ac}du^adu^c)$$

with unknown functions $h_{ac} = h_{ac}(\varrho, x^A, u^b)$ not depending on the coordinates v^1, \dots, v^p . Then, computations completely analogous to [15] prove:

Theorem 2.1. *Let g_H be a pp-wave metric as in (2.1) defined by functions $H_{ac} = H_{ac}(x^A, u^b)$. Then the metric $\tilde{g}_{H,h}$ in (2.3) is Ricci-flat if and only if each of the unknown functions $h_{bd} = h_{bd}(\varrho, x^A, u^b)$ satisfies the linear second order linear PDE*

$$(2.4) \quad 2\varrho h''_{bd} + (2-n)h'_{bd} - \Delta(H_{bd} + h_{bd}) = 0,$$

where the prime denotes the ϱ -differentiation. In particular, the metric $\tilde{g}_{H,h}$ is an ambient metric for the conformal class $[g_H]$ given by the pp-wave g_H if, in addition, the initial conditions

$$h_{bd}(\varrho, x^A, u^b)|_{\varrho=0} = 0$$

are satisfied.

In [15] we have seen how equation (2.4) can be solved by standard power series expansion, noticing that its indicial exponents are $s = 0$ and $s = n/2$. Here we give a more general existence result which includes non-analytic solutions and reflects the non-uniqueness of the ambient metric in even dimensions by explicitly showing the ambiguity at order $n/2$.

Theorem 2.2. *Let $\Delta = \delta^{AB}\partial_A\partial_B$ be the flat Laplacian in $(n-2p)$ dimensions and $H = H(x^A, u^b)$.*

(i) *When n is odd, the most general solutions $h = h(\varrho, x^A, u^b)$ to equation*

$$(2.5) \quad 2\varrho h'' + (2-n)h' - \Delta(H + h) = 0,$$

with $h(\varrho) \rightarrow 0$ when $\varrho \downarrow 0$ are given as

$$(2.6) \quad h = \sum_{k=1}^{\infty} \frac{\Delta^k H}{k! \prod_{i=1}^k (2i-n)} \varrho^k + \varrho^{n/2} \left(\alpha + \sum_{k=1}^{\infty} \frac{\Delta^k \alpha}{k! \prod_{i=1}^k (2i+n)} \varrho^k \right),$$

where $\alpha = \alpha(x^A, u^b)$ is an arbitrary function of its variables. In particular, when H is analytic, a unique solution that is analytic in ϱ in a neighbourhood of $\varrho = 0$ with $h(0) = 0$ is given by $\alpha \equiv 0$.

(ii) *When $n = 2s$ is even, the most general solutions h to equation (2.5) with $h(\varrho) \rightarrow 0$ when $\varrho \downarrow 0$ are given by*

$$(2.7) \quad \begin{aligned} h = & \sum_{k=1}^{s-1} \frac{\Delta^k H}{k! \prod_{i=1}^k (2i-n)} \varrho^k + \varrho^s \left(\alpha + \sum_{k=1}^{\infty} \frac{\Delta^k \alpha}{k! \prod_{i=1}^k (2i+n)} \varrho^k \right) \\ & + c_n \varrho^s \left(\sum_{k=0}^{\infty} (\log(\varrho) - q_k) \frac{\Delta^{s+k} H}{k! \prod_{i=1}^k (2i+n)} \varrho^k \right) \\ & + c_n \varrho^s Q * \sum_{k=0}^{\infty} \frac{\Delta^{s+k} H}{k! \prod_{i=1}^k (2i+n)} \varrho^k, \end{aligned}$$

where $\alpha = \alpha(x^A, u^b)$ and $Q = Q(x^A, u^b)$ are arbitrary functions of their variables, $$ denotes the convolution of two functions with respect to the x^A -variables, and the constants are given*

as

$$c_n = -\frac{1}{(s-1)! \prod_{i=0}^{s-1} (2i-n)}, \quad q_0 = 0, \quad q_k = \sum_{i=1}^k \frac{n+4i}{i(n+2i)}, \quad \text{for } k = 1, 2, \dots$$

In particular, only when $\Delta^s H \equiv 0$ are there solutions that are analytic in ϱ in a neighbourhood of $\varrho = 0$ and with $h(0) = 0$. These solutions are not unique but are parametrized by analytic functions α .

Proof. Let h be a solution to (2.5) with initial conditions $h(\varrho, x^A, u^b)|_{\varrho=0} \equiv 0$. Then $F = H + h$ is a solution to the homogeneous equation

$$(2.8) \quad \varrho F'' + (1 - \frac{n}{2})F' - \frac{1}{2}\Delta F = 0,$$

with the initial condition $F(\varrho, x^A, u^b)|_{\varrho=0} = H(x^A, u^b)$. Let \hat{F} and \hat{H} denote the Fourier transforms with respect to the x^A -variables of F and H . Recall that $\widehat{\Delta F} = -\|x\|^2 \hat{F}$, we easily find the Fourier transform of equation (2.8) to be

$$(2.9) \quad \varrho \hat{F}'' + (1 - \frac{n}{2})\hat{F}' + \frac{\|x\|^2}{2}\hat{F} = 0,$$

with initial condition $\hat{F}(\varrho, x^A, u^b)|_{\varrho=0} = \hat{H}$. For each point $(x^A, u^b) \in \mathbb{R}^{n-p}$, which we fix for the moment, this is an ODE in ϱ whose solutions can be determined by the Frobenius method (see for example [6, Section 6.3.3]). The indicial polynomial of (2.9) is $\lambda(\lambda-1) - (\frac{n}{2}-1)\lambda$ and hence has zeros 0 and $\frac{n}{2}$, which forces us to distinguish the cases when n is odd and n is even.

For n odd the general solution to (2.9) is given by

$$(2.10) \quad \hat{F} = a\varrho^{n/2}F_1 + bF_2,$$

for arbitrary constants a and b , and with F_1 and F_2 given by

$$(2.11) \quad F_1(x^A, \varrho) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k \|x\|^{2k}}{k! \prod_{i=1}^k (2i+n)} \varrho^k$$

$$(2.12) \quad F_2(x^A, \varrho) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k \|x\|^{2k}}{k! \prod_{i=1}^k (2i-n)} \varrho^k.$$

Hence, when we vary (x^A, u^b) we get that the general solution to (2.9) is given by

$$(2.13) \quad \hat{F} = \alpha \varrho^{n/2} F_1 + \beta F_2,$$

where $\alpha = \alpha(x^A, u^b)$ and $\beta = \beta(x^A, u^b)$ are arbitrary functions of their variables. The initial conditions $\hat{F}|_{\varrho=0} = \hat{H}$, or more precisely $\hat{F} \rightarrow \hat{H}$ if $\varrho \downarrow 0$, imposes $\beta = \hat{H}$. Hence, we obtain

$$\begin{aligned} \hat{F} &= \sum_{k=0}^{\infty} \frac{(-1)^k \|x\|^{2k} \hat{H}}{k! \prod_{i=1}^k (2i-n)} \varrho^k + \sum_{k=0}^{\infty} \frac{(-1)^k \|x\|^{2k} \alpha}{k! \prod_{i=1}^k (2i+n)} \varrho^k \\ &= \sum_{k=0}^{\infty} \frac{\widehat{\Delta^k H}}{k! \prod_{i=1}^k (2i-n)} \varrho^k + \sum_{k=0}^{\infty} \frac{\widehat{\Delta^k \alpha}}{k! \prod_{i=1}^k (2i+n)} \varrho^k, \end{aligned}$$

taking into account that $(-1)^k \|x\|^{2k} \hat{H} = \widehat{\Delta^k H}$ and denoting by $\check{\alpha}$ the inverse Fourier transform of α . Applying the inverse Fourier transform to this expression gives us

$$F = \sum_{k=0}^{\infty} \frac{\Delta^k H}{k! \prod_{i=1}^k (2i-n)} \varrho^k + \sum_{k=0}^{\infty} \frac{\Delta^k \check{\alpha}}{k! \prod_{i=1}^k (2i+n)} \varrho^k,$$

and thus the formula in the case n odd.

If $n = 2s$ is even, the general solution to (2.9) is given by

$$(2.14) \quad \hat{F} = \alpha \varrho^s F_1 + \beta \left(c_n (-1)^s \|x\|^n \varrho^s \log(\varrho) F_1 + F_3 \right)$$

where $\alpha = \alpha(x^A, u^b)$ and $\beta = \beta(x^A, u^b)$ are arbitrary functions of their variables, c_n is the constant defined in the statement of the theorem, F_1 is defined in (2.11) and F_3 can be computed as

$$F_3 = 1 + \sum_{k=1}^{s-1} \frac{(-1)^k \|x\|^{2k}}{k! \prod_{i=1}^k (2i - n)} \varrho^k - c_n (-1)^s \|x\|^n \varrho^s \left(\sum_{k=1}^{\infty} \frac{(-1)^k \|x\|^{2k}}{k! \prod_{i=1}^k (2i + n)} \hat{q}_k \varrho^k \right).$$

Here $\hat{q}_0 = \hat{q}_0(x^A, u^b)$ is an arbitrary function of the x^A 's and the u^c 's and the remaining \hat{q}_k are defined as

$$\hat{q}_k = \hat{q}_0 + \sum_{i=1}^k \frac{n + 4i}{i(n + 2i)}, \quad \text{for } k = 1, 2, \dots$$

Again, the initial conditions $\hat{F}|_{\varrho=0} = \hat{H}$, or more precisely $\hat{F} \rightarrow \hat{H}$ when $\varrho \downarrow 0$, imposes $\beta = \hat{H}$. Applying the inverse Fourier transform to our solutions \hat{F} , and taking into account that $(-1)^k \|x\|^{2k} \hat{H} = \widehat{\Delta^k H}$ and that the convolution satisfies $\widehat{Q * \Delta H} = 2\pi^{\frac{n-2}{2}} \hat{Q} \widehat{\Delta H}$, yields the formula in the theorem with the function Q in the theorem given by the inverse Fourier transform of $2\pi^{\frac{n-2}{2}} \hat{q}_0$. \square

Remark 2.1. Theorems 2.1 and 2.2 have the following consequences:

- (i) For n odd, the solutions h_{ac} given in Theorem 2.2 by analytic functions H_{ac} defining the pp-wave metric g_H in (2.1) provide us with the unique analytic ambient metric

$$(2.15) \quad \tilde{g}_H = 2d(\varrho t)dt + t^2 \left(g + 2 \left(\sum_{k=1}^{\infty} \frac{\Delta^k H_{ac}}{k! \prod_{i=1}^k (2i - n)} \varrho^k \right) du^a du^c \right)$$

for the conformal class of g_H .

- (ii) Theorem 2.2 also shows that, when n is odd, the ambiguity introduced by the arbitrary function α gives only *non-analytic* solutions, as guaranteed by the uniqueness statement in the Fefferman-Graham result. In contrast, when n is even, the ambiguity coming from the function α adds an analytic part to a solution and, in case of $\Delta^{n/2} H_{ac} = 0$, gives new *analytic* ambient metrics.

We conclude this section by giving an example of a Lorentzian pp-wave (i.e., with $p = 1$), which, in even dimensions, does not satisfy the sufficient condition $\Delta^{n/2} H = 0$ for our ansatz to give an analytic ambient metric, but which however admits non-analytic ambient metrics.

Example 2.1. For $k \in \mathbb{R}$ we consider the n -dimensional Lorentzian pp-wave

$$g = 2dudv + e^{2kx^1} du^2 + \sum_{A=1}^{n-2} (dx^A)^2,$$

i.e., with $H = \frac{1}{2}e^{2kx^1}$, which is a product of a 3-dimensional pp-wave and the flat Euclidean space of dimension $n - 3$. When $k = 0$ this is the flat metric which has a flat ambient metric. Hence we assume $k \neq 0$ from now on. Note that, when n even, this metric does not satisfy the condition $\Delta^{n/2} H = 0$, which ensures the existence of an analytic ambient metric. However, for finding an ambient metric (non-analytic when n even) we make the ansatz with $h = \frac{1}{2}(\phi(\varrho) - 1)e^{2kx^1}$ for a function ϕ of ϱ with $\phi(0) = 1$ and set

$$\tilde{g} = 2d(\varrho t)dt + t^2(g + 2hdu^2) = 2d(\varrho t)dt + t^2 \left(2dudv + \phi(\varrho)e^{2kx^1} du^2 + \sum_{i=1}^{n-2} (dx^i)^2 \right).$$

According to Theorem 2.1 this metric is Ricci flat if

$$2\varrho\phi'' + (2-n)\phi' - 4k^2\phi = 0,$$

and the initial condition is $\phi(0) = 1$. Of course, this equation can be solved using the power series techniques in Theorem 2.2 exhibiting the difference between n odd and even. But, for example, when $n = 4$ this equation becomes

$$\varrho\phi'' - \phi' - 2k^2\phi = 0.$$

The general solution with $\phi(0) = 1$ is given in closed form using the modified Bessel functions I_2 and K_2 as

$$\phi(\varrho) = 4k^2\varrho K_2(2k\sqrt{2\varrho}) + C\varrho I_2(2k\sqrt{2\varrho}),$$

where C is an arbitrary constant. This solution is not analytic at $\varrho = 0$, since the function $\varrho \mapsto \varrho K_2(2k\sqrt{2\varrho})$ fails to have a bounded second derivative at $\varrho = 0$. The function $\varrho \mapsto \varrho I_2(2k\sqrt{2\varrho})$ is analytic and the constant C is a remnant of the non-uniqueness. In contrast, when $n = 5$ the equations becomes

$$\varrho\phi'' - \frac{3}{2}\phi' - 2\phi = 0.$$

The general solution with $\phi(0) = 1$ is given by

$$\begin{aligned} \phi(\varrho) = & \cosh(2k\sqrt{2\varrho}) - 2k\sqrt{2}\sinh(2k\sqrt{2\varrho}) + \frac{8k^2}{3}\varrho \cosh(2k\sqrt{2\varrho}) \\ & + 3C \left(\sinh(2k\sqrt{2\varrho}) - 2k\sqrt{2}\cosh(2k\sqrt{2\varrho}) + \frac{8k^2}{3}\varrho \sinh(2k\sqrt{2\varrho}) \right), \end{aligned}$$

with an arbitrary constant C . The unique analytic solution is obtained by $C = 0$.

2.3. On the ambient holonomy for conformal pp-waves. We conclude our observations about conformal pp-waves by describing the holonomy of their ambient metrics. In this section we describe the possible holonomy for ambient metrics of conformal pp-waves.

Remark 2.2 (Terminology). In this and some of the following sections, we will determine the holonomy algebras of semi-Riemannian manifolds $(\widetilde{M}, \widetilde{g})$ at some point $p \in \widetilde{M}$. However, since the holonomy algebras at different points are conjugated to each other in $\mathbf{SO}(r, s)$, we will not make the point explicit in our notation. For a parallel tensor field Φ , i.e. with $\widetilde{\nabla}\Phi = 0$, we will use the terminology *stabilizer of Φ* by which we mean all linear maps $H \in \mathfrak{so}(T_p\widetilde{M}, g_p)$ that act trivially on $\Phi|_p$, i.e. $H \cdot \Phi|_p = 0$. If $\widetilde{\nabla}\Phi = 0$, then the holonomy algebra is contained in the stabilizer of Φ .³ Moreover, we say that a distribution \mathcal{V} is parallel if it is invariant under parallel transport, or equivalently if $\widetilde{\nabla}_X V \in \Gamma(\mathcal{V})$ for all $X \in T\widetilde{M}$ and all $V \in \Gamma(\mathcal{V})$. The *stabilizer of \mathcal{V}* consists of all linear maps in $\mathfrak{so}(T_p\widetilde{M}, g_p)$ that leave $\mathcal{V}|_p$ invariant. Again, if there is a parallel distribution, then the holonomy algebra is contained in its stabilizer.

First we note that a pp-wave (M, g) as defined in (2.1) admits a parallel null distribution \mathcal{V} spanned by the parallel vector fields $\frac{\partial}{\partial v^a}$, for $a = 1, \dots, p$. Moreover, the image of the Ricci endomorphism is contained in \mathcal{V} . Then, on the other hand, the results in [17, Theorem 1.1], in the case $p = 1$, and the generalization in [19, Theorem 1] imply that the normal conformal tractor bundle admits a totally null subbundle of rank $p + 1$ that is parallel for the normal conformal tractor connection of $[g]$, i.e., its fibres are invariant under the conformal holonomy. On the other hand, in the case of odd-dimensional analytic pp-waves, the general theory in [12] ensures that

³Note that the converse is not true: a tensor whose stabilizer contains the holonomy algebra, must not be parallel. This can be easily understood in the example of a parallel vector field. If one multiplies the parallel vector field by a function, the resulting vector field is no longer parallel, however still invariant under the holonomy at each point.

parallel objects of the tractor connection carry over to parallel objects of the analytic ambient metric. Hence, for an odd dimensional analytic pp-wave, the holonomy of the analytic ambient metric in (2.15) admits an invariant totally null plane of rank $p + 1$. However, in the next theorem we verify and strengthen this result by directly determining the holonomy of general metrics of the form (2.3). We will prove the following result in the case of $p = 1$ and in Lorentzian signature but it clearly generalises to larger p and other signatures in an obvious way.

Theorem 2.3. *Let $F = F(\varrho, u, x^1, \dots, x^{n-2})$ be a smooth function on \mathbb{R}^n . Then the metric*

$$\tilde{g}_F = 2d(t\varrho)dt + t^2 \left(2du(dv + F du) + \delta_{AB} dx^A dx^B \right)$$

on $\tilde{M} = \mathbb{R}^{n+1} \times \mathbb{R}_{>0}$ satisfies the following properties:

- (i) *The distribution $\tilde{\mathcal{V}} = \text{span}(\partial_v, \partial_\varrho)$ is parallel, i.e., it is invariant under parallel transport.*
- (ii) *Let $\mathcal{V} = \mathbb{R} \cdot \partial_v$ be the distribution of null lines spanned by ∂_v and \mathcal{V}^\perp be the distribution of vectors in $T\tilde{M}$ that are orthogonal to ∂_v , i.e., $\mathcal{V}^\perp = \text{span}(\partial_v, \partial_\varrho, \partial_1, \dots, \partial_{n-2}, \partial_t)$. Then the curvature \tilde{R} of \tilde{g}_F satisfies*

$$\tilde{R}(U, V)Y = 0, \quad \text{for all } U, V \in \mathcal{V}^\perp \text{ and } Y \in T\tilde{M}.$$

- (iii) *The holonomy algebra $\mathfrak{hol}(\tilde{g}_F)$ of \tilde{g}_F is contained in*

$$\left(\mathfrak{sl}_2\mathbb{R} \ltimes (\mathbb{R}^2 \otimes \mathbb{R}^{n-2}) \right) \oplus \mathbb{R} = \left\{ \begin{pmatrix} X & Z & aJ \\ 0 & 0 & -Z^\top \\ 0 & 0 & -X^\top \end{pmatrix} \mid \begin{array}{l} X \in \mathfrak{sl}_2\mathbb{R}, \\ Z \in \mathbb{R}^2 \otimes \mathbb{R}^{n-2}, \\ a \in \mathbb{R} \end{array} \right\},$$

$$\text{where } J \text{ is the } 2 \times 2\text{-matrix } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Proof. Let $\tilde{\nabla}$ be the Levi-Civita connection of \tilde{g}_F . Then a straightforward computation reveals that

$$\begin{aligned} \tilde{\nabla}\partial_v &= \frac{1}{t}\partial_v \otimes dt - \partial_\varrho \otimes du, \\ \tilde{\nabla}\partial_\varrho &= \frac{1}{t}\partial_\varrho \otimes dt + F_\varrho \partial_v \otimes du, \\ \tilde{\nabla}\partial_A &= \frac{1}{t}\partial_A \otimes dt + F_A \partial_v \otimes du - \partial_\varrho \otimes dx^A, \quad A = 1, \dots, n-2, \end{aligned} \tag{2.16}$$

in which $F_A = \partial_A(F)$, and $F_\varrho = \partial_\varrho(F)$. The first two equations show that the distribution $\tilde{\mathcal{V}}$ is invariant under parallel transport, and also that, in general, there is no invariant null line in $\tilde{\mathcal{V}}$. Moreover, it allows us to show that the curvature satisfies $\tilde{R}(X, Y)\partial_v = 0$ for all $X, Y \in T\tilde{M}$,

$$\begin{aligned} \tilde{R}(\partial_A, \partial_u)\partial_\varrho &= \tilde{R}(\partial_\varrho, \partial_u)\partial_i = F_{A\varrho}\partial_v \\ \tilde{R}(\partial_\varrho, \partial_u)\partial_\varrho &= F_{\varrho\varrho}\partial_v, \\ \tilde{R}(\partial_u, \partial_B)\partial_A &= -(\delta_{AB}F_\varrho + F_{AB})\partial_v, \end{aligned}$$

and that all other terms of the form $\tilde{R}(X, Y)\partial_\varrho$ and $\tilde{R}(X, Y)\partial_i$ are zero, unless the symmetry of \tilde{R} prevents this. This shows that the image of $\tilde{\mathcal{V}}^\perp$ under $\tilde{R}(X, Y)$ is contained in $\mathbb{R} \cdot \partial_v$. The symmetries of the curvature then imply part (ii) of the Theorem.

For the last part, first we note that, since $\tilde{\mathcal{V}}$ is parallel, the holonomy algebra of \tilde{g} is contained in the stabilizer in $\mathfrak{so}(2, n)$ of the totally null plane $\tilde{\mathcal{V}}$, which is equal to

$$(2.17) \quad (\mathfrak{gl}_2\mathbb{R} \oplus \mathfrak{so}(n-2)) \ltimes (\mathbb{R}^2 \otimes \mathbb{R}^{n-2} \oplus \mathbb{R}) = \left\{ \begin{pmatrix} X & Z & aJ \\ 0 & S & -Z^\top \\ 0 & 0 & -X^\top \end{pmatrix} \mid \begin{array}{l} X \in \mathfrak{gl}_2\mathbb{R}, \ a \in \mathbb{R} \\ Z \in \mathbb{R}^2 \otimes \mathbb{R}^{n-2}, \\ S \in \mathfrak{so}(n-2) \end{array} \right\}.$$

Moreover, the equations (2.16) show that the 2-form $\mu = tdt \wedge du$ is parallel with respect to the Levi-Civita connection. Hence, the projection of the holonomy to $\mathfrak{gl}_2\mathbb{R}$ actually lies in $\mathfrak{sl}_2\mathbb{R}$, i.e., $X \in \mathfrak{sl}_2\mathbb{R}$. Note that $\mathbb{R} \cdot J$ commutes with all of $(\mathfrak{sl}_2\mathbb{R} \oplus \mathfrak{so}(n-2)) \ltimes (\mathbb{R}^2 \otimes \mathbb{R}^{n-2})$, so the holonomy reduces to

$$\left(\mathfrak{sl}_2\mathbb{R} \oplus \mathfrak{so}(n-2) \right) \ltimes (\mathbb{R}^2 \otimes \mathbb{R}^{n-2}) \oplus \mathbb{R}.$$

Finally, to show that the elements in the holonomy algebra have no $\mathfrak{so}(n-2)$ -component, i.e., that $S = 0$ in (2.17), we fix a point $p \in \widetilde{M}$ and use the Ambrose-Singer Holonomy Theorem to show that $\mathfrak{hol}_p(\widetilde{g})$ maps $\widetilde{\mathcal{V}}^\perp|_p$ to $\widetilde{\mathcal{V}}|_p$. Indeed, let $V \in \widetilde{\mathcal{V}}^\perp|_p$ be in the fibre of $\widetilde{\mathcal{V}}^\perp$ at p . Let $V_q = \mathcal{P}_\gamma(V)$ denote the parallel transport of V along a curve that ends at $q \in \widetilde{M}$. By the invariance of the distribution $\widetilde{\mathcal{V}}^\perp$ we have $V_q \in \widetilde{\mathcal{V}}^\perp|_q$, however applying the curvature at q , $\widetilde{R}_q(X, Y)$ for $X, Y \in T_q\widetilde{M}$, gives

$$\widetilde{R}_q(X, Y)V_q \in \mathcal{V}|_q \subset \widetilde{\mathcal{V}}|_q,$$

by the above formulae. Since also $\widetilde{\mathcal{V}}$ is invariant under parallel transport, we obtain

$$\mathcal{P}_\gamma^{-1} \circ R(X, Y) \circ \mathcal{P}_\gamma(V) \in \widetilde{\mathcal{V}}|_p,$$

for every $V \in \widetilde{\mathcal{V}}^\perp|_p$. Then the Ambrose-Singer Theorem provides us with the third statement. \square

For a conformal pp-wave defined by a function H , this theorem for $F = H + h$ gives an upper bound for the holonomy algebra of the analytic ambient metric \widetilde{g} in (2.15) in the cases when it exists, e.g., when n odd or when $\Delta^{\frac{n}{2}}H = 0$. This upper bound is an improvement of the result in [15, Corollary 2]. The special structure of the ambient metric, in particular its Ricci flatness, might reduce the holonomy further. Indeed, for special conformal pp-waves in Lorentzian signature, such as plane waves or Cahen-Wallach spaces, the ambient holonomy reduces further:

Proposition 2.1. *Let $(S_{AB})_{i,j=1}^{n-2}$ be a symmetric matrix of functions $S_{AB} = S_{AB}(u)$ with trace $S = S(u)$ and let*

$$(2.18) \quad g = 2du dv + 2(S_{AB}(u) x^A x^B) du^2 + \delta_{AB} dx^A dx^B$$

be the corresponding Lorentzian plane wave metric on \mathbb{R}^n . If $f = f(u)$ is a solution to the ODE

$$f'' = (f')^2 - \frac{2}{n-2}S,$$

then the metric $\hat{g} = e^{2f}g$ is Ricci-flat. Moreover, the metric

$$\widetilde{g} = 2d(t\varrho)dt + t^2 e^{2f(u)}g$$

is an ambient metric for the conformal class $[g]$ and admits two parallel null vector fields $\frac{1}{t}\partial_\varrho$ and $\frac{1}{t}(\partial_v + h\partial_\varrho)$, where $h = h(u)$ is a solution to $h' = e^{2f}$. Consequently, the holonomy algebra of \widetilde{g} is contained in $\mathbb{R}^2 \otimes \mathbb{R}^{n-2}$.

This proposition is a generalisation of the corresponding statements in [14] about the conformal holonomy of plane waves. Its proof follows from straightforward computations and the well-known formula (1.4) for the ambient metric of Einstein metrics in the case of $\Lambda = 0$. A class of examples to which this proposition applies are the symmetric Cahen-Wallach spaces for which the metric (2.18) is defined by a constant symmetric matrix S_{AB} .

3. GENERIC 2-DISTRIBUTIONS IN DIMENSION 5 AND THEIR AMBIENT METRICS

3.1. Conformal structures and generic rank 2 distributions in dimension 5. The examples of 5-dimensional conformal structures considered in Ref. [16] belong to a wider class of conformal metrics naturally associated with *generic rank 2 distributions in dimension 5*. The correspondence between these distributions and conformal structures with signature $(2, 3)$ is explained in detail in [20]. Here we recall this correspondence briefly for a particular subclass of these distributions:

Associated with a differential equation

$$z' = F(x, y, p, q, z),$$

for real functions $y = y(x)$ and $z = z(x)$, where $p = y'$, $q = y''$, there is a 5-manifold M parametrized by (x, y, p, q, z) , and a distribution

$$(3.1) \quad \mathcal{D} = \text{Span}(\partial_q, \partial_x + p \partial_y + q \partial_p + F \partial_z).$$

The distribution is *generic* if $F_{qq} \neq 0$. For generic distributions the fundamental differential invariants of Cartan [5] are in one-to-one correspondence with *conformal* invariants of a certain conformal class $[g_{\mathcal{D}}]$ of metrics of signature $(3, 2)$ on M . In this section we will consider distributions and the corresponding conformal classes defined by

$$(3.2) \quad F = q^2 + f(x, y, p) + h(x, y)z,$$

where f and h are smooth functions of their variables. For such F we denote the distribution \mathcal{D} by $\mathcal{D}_{f,h}$. The conformal class $[g_{\mathcal{D}_{f,h}}]$ may be represented by a metric

$$g_{\mathcal{D}_{f,h}} = 2\theta^1\theta^5 - 2\theta^2\theta^4 + (\theta^3)^2,$$

where the co-frame θ^i is given by

$$(3.3) \quad \begin{aligned} \theta^1 &= dy - p dx, \\ \theta^2 &= dz - (q^2 + f + hz) dx - \frac{\sqrt{2}}{2} q \theta^3, \\ \theta^3 &= 2\sqrt{2}(dp - q dx), \\ \theta^4 &= 3dx, \\ \theta^5 &= \frac{\sqrt{2}h}{2} \theta^3 - 6dq + 3(2hq + f_p) dx + \frac{1}{10} (9f_{pp} + 4h^2 - 6(ph_y + h_x)) \theta^1, \end{aligned}$$

and where f_p denotes the partial derivative $\partial_p(f)$, $h_x = \partial_x(h)$, etc.

In the examples we gave in [16], the function f was just a polynomial of degree 6 in p , and $h \equiv b$ was a constant. We were able to show that for a generic choice of b and a generic choice of the coefficients of the polynomial defining f , the ambient metric had the full exceptional Lie group \mathbf{G}_2 as holonomy, thus obtaining an 8-parameter family of \mathbf{G}_2 -metrics. This stimulated further research in [12], where the authors showed that the conformal structures $[g_{\mathcal{D}}]$ associated with analytic generic rank 2-distributions in dimension 5, the holonomy of the corresponding analytic ambient metric is *always* contained in \mathbf{G}_2 and *generically* equal to \mathbf{G}_2 . Motivated by this result, we now produce more explicit examples of ambient metrics for the conformal structures associated with distributions $\mathcal{D}_{f,h}$ with f and h defining F as in (3.2). As a byproduct we will obtain a large class of explicit metrics with holonomy equal to \mathbf{G}_2 , extending significantly our examples in [16].

3.2. An ansatz for the ambient metric, analytic solutions, and \mathbf{G}_2 -ambient metrics.

In this section we will find explicit formulae for all the Fefferman-Graham ambient metrics for conformal classes $(M, [g_{\mathcal{D}_{f,h}}])$, with distribution $\mathcal{D}_{f,h}$ related to $F = q^2 + f(x, y, p) + h(x, y)z$ via (3.1). Since conformal structures $[g_{\mathcal{D}_{f,h}}]$ associated with distributions $\mathcal{D}_{f,h}$ are defined in *odd* dimension $n = 5$, the result by Fefferman-Graham implies that there is a unique analytic ambient metric $\tilde{g}_{\mathcal{D}_{f,h}}$ for each pair of *analytic* functions f and h . In order to find this ambient metric,

we start with the following observation which parallels what we found previously for pp-waves: A direct computation shows that the Schouten tensor for the class $[g_{\mathcal{D}_{f,h}}]$ has the form

$$P = \alpha (\theta^1)^2 + 2\beta \theta^1 \theta^4 + \gamma (\theta^4)^2,$$

with θ^1 and θ^4 as in (3.3), and the functions α, β and γ determined by f and h and their derivatives:

$$(3.4) \quad \alpha = -\frac{3}{80}f_{pppp}, \quad \beta = -\frac{1}{80}(f_{ppp} + 6h_y), \quad \gamma = -\frac{1}{15}\left(\frac{1}{2}(h_x + ph_y) + \frac{1}{12}f_{pp} - \frac{1}{3}h^2\right)$$

Hence, as for pp-waves, we make an *ansatz* for the ambient metric $\tilde{g}_{\mathcal{D}_{f,h}}$ in which $g_{\mathcal{D}_{f,h}}(x^i, \varrho)$ assumes a similar form to P . Explicitly, we make the following ansatz for $\tilde{g}_{\mathcal{D}_{f,h}}$:

$$(3.5) \quad \tilde{g}_{\mathcal{D}_{f,h}} = 2dtd(\varrho t) + t^2 g_{\mathcal{D}_{f,h}}(x^i, \varrho) = 2dtd(\varrho t) + t^2 (g_{\mathcal{D}_{f,h}} + A(\theta^1)^2 + 2B\theta^1\theta^4 + C(\theta^4)^2),$$

with *unknown* functions $A = A(x, y, p, \varrho)$, $B = B(x, y, p, \varrho)$ and $C = C(x, y, p, \varrho)$. Miraculously, for this ansatz, the equations for $\text{Ric}(\tilde{g}_{\mathcal{D}_{f,h}}) = 0$ form a system of PDEs which are *linear* in the unknowns A, B, C :

Theorem 3.1. *The metric $\tilde{g}_{\mathcal{D}_{f,h}}$ in (3.5) is an ambient metric for the conformal class $(M, [g_{\mathcal{D}_{f,h}}])$ if and only if the unknown functions $A = A(x, y, p, \varrho)$, $B = B(x, y, p, \varrho)$ and $C = C(x, y, p, \varrho)$ satisfy the initial conditions $A|_{\varrho=0} \equiv 0$, $B|_{\varrho=0} \equiv 0$, $C|_{\varrho=0} \equiv 0$, and the following system of PDEs:*

$$(3.6) \quad \begin{aligned} LA &= \frac{9}{40}f_{pppp} \\ LB &= -\frac{1}{36}A_p + \frac{3}{40}f_{ppp} + \frac{9}{20}h_y \\ LC &= -\frac{1}{18}B_p + \frac{1}{324}A + \frac{1}{30}f_{pp} - \frac{2}{15}h^2 + \frac{1}{5}(ph_y + h_x), \end{aligned}$$

where the linear operator L is given by

$$L = 2\varrho \frac{\partial^2}{\partial \varrho^2} - 3 \frac{\partial}{\partial \varrho} - \frac{1}{8} \frac{\partial^2}{\partial p^2}.$$

The proof of this theorem proceeds by directly computing the Ricci-tensor $\text{Ric}(\tilde{g}_{\mathcal{D}_{f,h}})$, using the formulae (3.4) for the Schouten tensor of $g_{\mathcal{D}_{f,h}}$.

In order to find general formal solutions for the linear system (3.7) we assume the power series expansions of the unknowns $A = A(x, y, p, \varrho)$, $B = B(x, y, p, \varrho)$ and $C = C(x, y, p, \varrho)$, i.e.,

$$A = \sum_{k=1}^{\infty} a_k(x, y, p) \varrho^k, \quad B = \sum_{k=1}^{\infty} b_k(x, y, p) \varrho^k, \quad C = \sum_{k=1}^{\infty} c_k(x, y, p) \varrho^k.$$

We describe the solutions in the following Theorem. The proof that these give all analytic solutions is analogous to the pp-wave case.

Theorem 3.2. *The most general solution to the linear system (3.6), vanishing at $\varrho = 0$ and being analytic in ϱ whenever f is analytic, is given by*

$$\begin{aligned} A &= \frac{3}{5} \sum_{k=1}^{\infty} \frac{(2k-1)(2k-3)}{2^{2k}(2k)!} \frac{\partial^{(2k+2)} f}{\partial p^{(2k+2)}} \varrho^k, \\ B &= -\frac{3}{20} \varrho h_y - \frac{1}{15} \sum_{k=1}^{\infty} \frac{(2k-1)(2k-3)(2k-5)}{2^{2k}(2k)!} \frac{\partial^{(2k+1)} f}{\partial p^{(2k+1)}} \varrho^k, \\ C &= \frac{2}{45} \varrho h^2 - \frac{1}{15} \varrho (ph_y + h_x) + \frac{2}{135} \sum_{k=1}^{\infty} \left(\frac{(k-3)(2k-1)(2k-3)(2k-5)}{2^{2k}(2k)!} \frac{\partial^{2k} f}{\partial p^{2k}} \right) \varrho^k. \end{aligned}$$

Our next goal is to determine explicitly the parallel 3-form that gives the reduction to \mathbf{G}_2 of the holonomy of the ambient metric in (3.5). Such a holonomy reduction imposes conditions on the connection and hence implies first order conditions on the metric coefficients. Therefore, in order to find the 3-form, we have to identify these first order conditions. Our approach will be as follows: From [12] we know that for analytic functions f and h the unique analytic ambient metric will admit a parallel 3-form. This means that the above analytic solutions to the second order system (3.6), which give us the unique analytic Ricci flat ambient metric, will in turn, via the induced holonomy reduction, be the solutions to some first order equations on the functions A , B and C . It turns out that the first order system we will find in this way actually implies the second order equations (3.6) as their integrability conditions.

Theorem 3.3. *Let $f = f(x, y, p)$ and $h = h(x, y)$ be two smooth functions. Then the formal solutions A , B and C of the second order system (3.6) solve the following first order system*

$$(3.7) \quad \begin{aligned} B_p &= \frac{5}{9}A - \frac{2}{9}\varrho A_\varrho \\ B_\varrho &= -\frac{1}{72}A_p - \frac{1}{40}f_{ppp} - \frac{3}{20}h_y \\ C_\varrho &= \frac{1}{648}A - \frac{1}{72}B_p - \frac{1}{90}f_{pp} + \frac{2}{45}h^2 - \frac{1}{15}(ph_y + h_x) \\ C_p &= \frac{1}{324}\varrho A_p + \frac{2}{3}B + \frac{1}{180}\varrho f_{ppp} + \frac{1}{30}\varrho h_y. \end{aligned}$$

Moreover, twice differentiable solutions A , B and C of this first order system (3.7) are solutions to the second order system (3.6).

Proof. It is a matter of checking that the formal solutions in Theorem 3.2 of the second order system (3.6) satisfy the first order system (3.7).

In order to derive the second order equations from the first order system, we first use the integrability condition arising from the first two equations in (3.7). We immediately compute

$$0 = B_{\varrho p} - B_{p\varrho} = \frac{1}{9} \left(L(A) - \frac{9}{40}f_{pppp} \right),$$

i.e., the first equation in (3.6) is exactly this integrability condition. Note that the integrability condition arising from the last two equations, $C_{p\varrho} - C_{\varrho p} = 0$, immediately follows from the first two equations in (3.7):

$$C_{p\varrho} - C_{\varrho p} = \frac{1}{648}A_p + \frac{1}{324}\varrho A_{p\varrho} + \frac{1}{60}f_{ppp} + \frac{1}{10}h_y + \frac{1}{72}B_{pp} + \frac{2}{3}B_\varrho = 0,$$

when substituting $B_{pp} = \frac{5}{9}A_p - \frac{2}{9}\varrho A_{p\varrho}$ and $B_\varrho = -\frac{1}{72}A_p - \frac{1}{40}f_{ppp} - \frac{3}{20}h_y$ in the last two terms. As the first derivatives of B and C in the variables p and ϱ are explicitly given in (3.7), one checks that the second equation in (3.6) can be derived directly from the first two equations in (3.7), whereas the third equation in (3.6) requires all four equations in (3.7) for its derivation. \square

Remark 3.1. In Section 3.4, when determining *all* formal solutions to the second order system (3.6), we will see that (3.6) has solutions, defined on $\varrho \geq 0$ and only twice differentiable which do not solve the first order system (3.7), see Remark 3.4.

Remark 3.2. Note that for a set of solutions A , B , C of (3.7), the functions B and C are uniquely determined by the function A and the initial conditions.

The first order system (3.7) enables us to find the parallel 3-form that defines the \mathbf{G}_2 -reduction of the ambient metric. To this end we define a null co-frame for the ambient metric (3.5) by

$$(3.8) \quad \begin{aligned} \omega^1 &= 9\sqrt{2}dt + t\sqrt{2}h\theta^4, & \omega^2 &= \theta^1, & \omega^3 &= -\frac{1}{9}th(\varrho dt + t d\varrho) - t^2\theta^2 + \frac{1}{2}t^2C\theta^4, \\ \omega^4 &= t\theta^3, & \omega^5 &= \theta^4, & \omega^6 &= \frac{1}{2}tA\theta^1 + t^2B\theta^4 + t^2\theta^5, & \omega^7 &= \frac{\sqrt{2}}{18}(\varrho dt + t d\varrho), \end{aligned}$$

with the θ^i 's given in the equations (3.3). We obtain:

Theorem 3.4. *Let $\mathcal{D}_{f,h}$ be a generic distribution on \mathbb{R}^5 defined by*

$$F = q^2 + f(x, y, p) + h(x, y)z,$$

where f and h are smooth functions of their variables, and let $[g_{\mathcal{D}_{f,h}}]$ be the corresponding conformal class. Let $A = A(x, y, p, \varrho)$, $B = B(x, y, p, \varrho)$ and $C = C(x, y, p, \varrho)$ be twice differentiable solutions of the first order system (3.7) associated to f and h , with initial conditions $A|_{\varrho=0} = B|_{\varrho=0} = C|_{\varrho=0} = 0$. Then, in the frame defined in (3.8), an ambient metric is given as

$$(3.9) \quad \tilde{g}_{\mathcal{D}_{f,h}} = 2\omega^1\omega^7 + 2\omega^2\omega^6 + 2\omega^3\omega^5 + (\omega^4)^2,$$

with a parallel 3-form

$$(3.10) \quad \Upsilon = 2\omega^{123} - \omega^{147} - \omega^{246} - \omega^{345} + \omega^{567},$$

where we use the standard notation $\omega^{ijk} = \omega^i \wedge \omega^j \wedge \omega^k$.

Moreover, if f is analytic and we take A , B and C to be the solutions of the system (3.7) given in Theorem 3.2 and being analytic in ϱ , then the metric in (3.9) is the unique ambient metric that is analytic in ϱ .

Proof. The Ricci-flatness of $\tilde{g}_{\mathcal{D}_{f,h}}$ as in (3.5) follows from Theorems 3.1 and 3.3. The covariant derivative $\tilde{\nabla}\Upsilon$ with respect to the Levi-Civita connection $\tilde{\nabla}$ of $\tilde{g}_{\mathcal{D}_{f,h}}$ in (3.5) contains derivatives of f and h as well as first derivatives of A , B and C . Computing $\tilde{\nabla}\Upsilon$ and substituting the first derivatives of A , B and C with the help of the first order system (3.7) shows that $\tilde{\nabla}\Upsilon = 0$. \square

Remark 3.3. Note that this theorem not only gives us the unique analytic solution to the Fefferman-Graham equations but, in principle, also applies also to possible non-smooth solutions to the first order system (3.7). More generally, leaving the context of conformal structures aside, it provides a tool for constructing metrics with holonomy contained in \mathbf{G}_2 from solutions of the first order system (3.7). Below in Theorem 3.5 we will see what one has to assume for the solutions in order to ensure that such metrics have holonomy equal to \mathbf{G}_2 .

For completeness, we recall that there is a one-to-one correspondence between a 3-form making the reduction from $\mathbf{SO}(4, 3)$ to \mathbf{G}_2 and a spinor. Using the formula governing this correspondence (see e.g. [13] or [16, pp. 430]) we now find the spinor ψ corresponding to the 3-form Υ . This needs some preparations:

Since the spinor ψ takes the most convenient form in an orthonormal frame, we pass from the null co-frame (ω^i) to a co-frame ξ^i given by

$$\begin{aligned} \xi^1 &= \omega^1 + \frac{1}{2}\omega^7, & \xi^2 &= \frac{\sqrt{2}}{2}\omega^2 + \frac{\sqrt{2}}{2}\omega^6, & \xi^3 &= \frac{\sqrt{2}}{2}\omega^3 + \frac{\sqrt{2}}{2}\omega^5, & \xi^4 &= \omega^4, \\ \xi^5 &= \frac{\sqrt{2}}{2}\omega^3 - \frac{\sqrt{2}}{2}\omega^5, & \xi^6 &= \frac{\sqrt{2}}{2}\omega^2 - \frac{\sqrt{2}}{2}\omega^6, & \xi^7 &= \omega^1 - \frac{1}{2}\omega^7. \end{aligned}$$

This co-frame is orthonormal for $\tilde{g}_{f,h}$,

$$\tilde{g}_{f,h} = \tilde{g}_{ij}\xi^i\xi^j = (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 + (\xi^4)^2 - (\xi^5)^2 - (\xi^6)^2 - (\xi^7)^2,$$

and the parallel 3-form (3.10) becomes

$$\Upsilon = \xi^{125} - \xi^{136} + \xi^{147} + \xi^{237} + \xi^{246} + \xi^{345} - \xi^{567}.$$

To find the formula for the spinor ψ , we need a representation of the Clifford algebra $\text{Cl}(4, 3)$. To this end we introduce real 8×8 matrices σ_i , called the σ -matrices, satisfying the relation

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 2\tilde{g}_{ij}\mathbb{I}_8.$$

They generate the Clifford algebra $\text{Cl}(4, 3)$ acting on the vector space \mathbb{R}^8 of real spinors. Following [16, pp. 430] we write the σ -matrices as⁴

$$\begin{aligned} \sigma_1 &= \begin{pmatrix} 0 & \gamma_1 \\ \gamma_1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & \gamma_3 \\ \gamma_3 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & \gamma_5 \\ \gamma_5 & 0 \end{pmatrix}, \quad \sigma_4 = \begin{pmatrix} \mathbb{I}_4 & 0 \\ 0 & -\mathbb{I}_4 \end{pmatrix} \\ \sigma_5 &= \begin{pmatrix} 0 & \gamma_2 \\ \gamma_2 & 0 \end{pmatrix}, \quad \sigma_6 = \begin{pmatrix} 0 & \gamma_4 \\ \gamma_4 & 0 \end{pmatrix}, \quad \sigma_7 = \begin{pmatrix} 0 & -\mathbb{I}_4 \\ \mathbb{I}_4 & 0 \end{pmatrix}. \end{aligned}$$

where

$$\begin{aligned} \gamma_1 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \\ \gamma_4 &= \begin{pmatrix} 0 & -\mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}. \end{aligned}$$

We identify the spinor bundle $\mathcal{S} \rightarrow \tilde{M}$ with a vector bundle over \tilde{M} , with fibre \mathbb{R}^8 on which the orthonormal basis e_i , dual to ξ^i (i.e. $\xi^j(e_i) = \delta_i^j$), acts via the *Clifford multiplication*, i.e., via the multiplication defined by

$$e_i \psi = \sigma_i \psi.$$

Then the lift of the Levi-Civita connection from \tilde{M} to the spinor bundle \mathcal{S} is given by

$$\tilde{\nabla} \psi = d\psi + \frac{1}{4} \tilde{\Gamma}^{kl} \sigma_k \sigma_l \psi,$$

where $\tilde{\Gamma}^{kl}$ are the Levi-Civita connection 1-forms for the ambient metric $\tilde{g}_{f,h}$ in the orthonormal co-frame ξ^i . Explicitly they are defined by

$$\tilde{\Gamma}^{kl}(X) = \tilde{g}(\tilde{\nabla}_X \xi^k, \xi^l).$$

With this notation, the spinor field ψ corresponding to the parallel 3-form Υ is as simple as

$$\psi = (0, 1, -1, 0, 1, 0, 0, -1)^\top.$$

One can easily check that it is parallel, $\tilde{\nabla} \psi = 0$, with respect to the connection $\tilde{\nabla}$, and hence confirms that the metric $\tilde{g}_{f,h}$ has holonomy in \mathbf{G}_2 .

Next, we address the important question as to when the conformal classes $[g_{\mathcal{D}_{f,h}}]$ determines an ambient metric with holonomy *equal to* \mathbf{G}_2 . The next theorem gives a very simple sufficient condition.

Theorem 3.5. *Let A , B and C be solutions to the first order system (3.7) for functions $f = f(x, y, p)$ and $h = h(x, y)$, and assume that $A_\varrho \neq 0$. Then the holonomy of the ambient metric*

$$\tilde{g}_{\mathcal{D}_{f,h}} = 2dtd(\varrho t) + t^2 g_{\mathcal{D}_{f,h}}(x^i, \varrho) = 2dtd(\varrho t) + t^2 (g_{\mathcal{D}_{f,h}} + A(\theta^1)^2 + 2B\theta^1\theta^4 + C(\theta^4)^2),$$

for the conformal class defined as in (3.2) by f and h has holonomy equal to \mathbf{G}_2 .

⁴In fact, the σ -matrices come, as a special case, from A. Trautman's inductive construction of real representations for $\text{Cl}(g)$ in case of metrics g with split signature. This construction can be found in [21].

Proof. Let X_i , $i = 1, \dots, 7$ be the vector fields dual to the 1-forms in (3.8). The algebra of derivations, represented by $(1, 1)$ -tensors, which preserve the 3-form (3.10) is given by

$$\begin{aligned}
h_1 &= X_3 \otimes \omega_2 - X_6 \otimes \omega_5, \\
h_2 &= X_3 \otimes \omega_1 - X_7 \otimes \omega_5, \\
h_3 &= X_3 \otimes \omega_4 - X_4 \otimes \omega_5 + X_6 \otimes \omega_1 - X_7 \otimes \omega_2, \\
h_4 &= X_2 \otimes \omega_5 - X_3 \otimes \omega_6 + 2X_4 \otimes \omega_1 + 2X_7 \otimes \omega_4, \\
h_5 &= X_1 \otimes \omega_5 - X_3 \otimes \omega_7 - 2X_4 \otimes \omega_2 + 2X_6 \otimes \omega_4, \\
h_6 &= X_2 \otimes \omega_1 - X_7 \otimes \omega_6, \\
h_7 &= X_1 \otimes \omega_1 - 2X_2 \otimes \omega_2 + X_3 \otimes \omega_3 - X_5 \otimes \omega_5 + 2X_6 \otimes \omega_6 - X_7 \otimes \omega_7, \\
h_8 &= X_2 \otimes \omega_2 - X_3 \otimes \omega_3 + X_5 \otimes \omega_5 - X_6 \otimes \omega_6, \\
h_9 &= X_1 \otimes \omega_2 - X_6 \otimes \omega_7, \\
h_{10} &= X_2 \otimes \omega_4 - X_4 \otimes \omega_6 - X_5 \otimes \omega_1 + X_7 \otimes \omega_3, \\
h_{11} &= X_1 \otimes \omega_4 - X_4 \otimes \omega_7 + X_5 \otimes \omega_2 - X_6 \otimes \omega_3, \\
h_{12} &= X_1 \otimes \omega_6 - X_2 \otimes \omega_7 + 2X_4 \otimes \omega_3 - 2X_5 \otimes \omega_4, \\
h_{13} &= X_2 \otimes \omega_3 - X_5 \otimes \omega_6, \\
h_{14} &= X_1 \otimes \omega_3 - X_5 \otimes \omega_7.
\end{aligned}$$

Because the 3-form Υ is parallel, the curvature tensor $R(X, Y)$ and its directional covariant derivatives of the curvature tensor lie in the span of these $(1, 1)$ -tensors. To prove the theorem, we have to check, conversely, that all the tensors h_i are in the span of the curvature tensor $R(X, Y)$ and its directional covariant derivatives. To this end set $\mathcal{H}_i = \text{span}\{h_1, h_2, \dots, h_i\}$. Then one easily calculates for the curvature and its derivatives

$$\begin{aligned}
R(X_3, X_4) &= \frac{1}{2} A_\varrho h_1, & \nabla_{X_2} h_1 &= -\frac{\sqrt{2}}{18t} h_2, \\
\nabla_{X_5} h_1 &= \frac{\sqrt{2}}{18t} h_3 \mod \mathcal{H}_2, & \nabla_{X_5} h_3 &= \frac{\sqrt{2}}{18t} h_4 \mod \mathcal{H}_3, \\
\nabla_{X_2} h_4 &= -9t \frac{\sqrt{2}}{2} A_\varrho h_5 \mod \mathcal{H}_4, & \nabla_{X_5} h_4 &= -\frac{\sqrt{2}}{6t} h_6 \mod \mathcal{H}_5, \\
\nabla_{X_5} h_5 &= -\frac{\sqrt{2}}{18t} h_7 \mod \mathcal{H}_6, & \nabla_{X_2} h_6 &= -9t \frac{\sqrt{2}}{2} A_\varrho h_8 \mod \mathcal{H}_7, \\
\nabla_{X_2} h_7 &= 27t \frac{\sqrt{2}}{2} A_\varrho h_9 \mod \mathcal{H}_8, & \nabla_{X_5} h_8 &= \frac{\sqrt{2}}{18t} h_{10} \mod \mathcal{H}_9, \\
\nabla_{X_2} h_9 &= \frac{\sqrt{2}}{18t} h_{11} \mod \mathcal{H}_{10}, & \nabla_{X_5} h_{11} &= -\frac{\sqrt{2}}{18t} h_{12} \mod \mathcal{H}_{11}, \\
\nabla_{X_5} h_{12} &= -\frac{\sqrt{2}}{6t} h_{13} \mod \mathcal{H}_{12}, & \nabla_{X_2} h_{13} &= -9t \frac{\sqrt{2}}{2} A_\varrho h_{14} \mod \mathcal{H}_{13}.
\end{aligned}$$

This establishes the theorem. \square

Let us compare this sufficient criterion with the one given in [16] for conformal classes defined by $h(x, y) \equiv b$ and $f(x, y, p) = \sum_{i=0}^6 a_i p^i$ with constants a_i and b : There it was shown that such a conformal class has ambient holonomy equal to \mathbf{G}_2 , if $a_3^2 + a_4^2 + a_4^2 + a_5^2 + a_6^2 \neq 0$. Clearly the case $f(x, y, p) = p^3$ is not covered by the new criterion in Theorem 3.5, since in this case we have $A \equiv 0$, but as soon as $a_4 \neq 0$ the new criterion applies.

3.3. Examples. Returning to Theorem 3.2, we see that if we want to have examples of ambient metrics that are *polynomial in ϱ* of order at most k we need to have $\frac{\partial^{(2k+2)}f}{\partial p^{(2k+2)}} \equiv 0$, i.e. the function $f = f(x, y, p)$ defining the distribution must be a polynomial in p of order no higher than $2k + 1$. On the other hand, as soon as we have $\frac{\partial^4 f}{\partial p^4} \neq 0$, and hence $A_\varrho \neq 0$, Theorem 3.5 ensures that the holonomy of the ambient metric is equal to \mathbf{G}_2 . Hence, combining the results of Theorems 3.1, 3.2 and 3.5 we can construct *explicit* signature $(4, 3)$ metrics with *full* \mathbf{G}_2 holonomy which are given in closed form, as polynomials in the variables p and ϱ , provided that the function $f = f(x, y, p)$ is a polynomial in p of order $k \geq 4$. Here is an example of such a full \mathbf{G}_2 holonomy metrics that are polynomial in p and ϱ :

Example 3.1. On $M = \{(x, y, z, p, q)\}$ let f_0, f_1, \dots, f_9 be arbitrary real analytic functions of the variables x and y . Define a function f of x, y and p as

$$f = f_0 + f_1 p + f_2 p^2 + f_3 p^3 + f_4 p^4 + f_5 p^5 + f_6 p^6 + f_7 p^7 + f_8 p^8 + f_9 p^9.$$

Then Theorem 3.2 implies that the ambient metric for $[g_{\mathcal{D}_{f,0}}]$ is

$$(3.11) \quad \tilde{g} = 2dt d(\varrho t) + t^2 \left(2\omega^1 \omega^5 - 2\omega^2 \omega^4 + (\omega^3)^2 + A(\omega^1)^2 + 2B\omega^1 \omega^4 + C(\omega^4)^2 \right),$$

where the ω^i 's are defined in (3.3) and the functions A, B and C given on \tilde{M} by

$$\begin{aligned} A &= \frac{63}{8}(f_8 + 9pf_9)\varrho^3 + \frac{27}{8}(f_6 + 7pf_7 + 28p^2 f_8 + 84p^3 f_9)\varrho^2 - \\ &\quad \frac{9}{5}(f_4 + 5pf_5 + 15p^2 f_6 + 35p^3 f_7 + 70p^4 f_8 + 126p^5 f_9)\varrho, \\ B &= -\frac{63}{256}f_9\varrho^4 - \frac{7}{64}(f_7 + 8pf_8 + 36p^2 f_9)\varrho^3 + \frac{1}{16}(f_5 + 6pf_6 + 21p^2 f_7 + 56p^3 f_8 + 126p^4 f_9)\varrho^2 - \\ &\quad \frac{3}{20}(f_3 + 4pf_4 + 10p^2 f_5 + 20p^3 f_6 + 35p^4 f_7 + 56p^5 f_8 + 84p^6 f_9)\varrho, \\ C &= \frac{7}{1152}(f_8 + 9pf_9)\varrho^4 + \frac{1}{360}(f_4 + 5pf_5 + 15p^2 f_6 + 35p^3 f_7 + 70p^4 f_8 + 126p^5 f_9)\varrho^2 + \\ &\quad \frac{1}{45}(2b^2 - f_2 - 3pf_3 - 6p^2 f_4 - 10p^3 f_5 - 15p^4 f_6 - 21p^5 f_7 - 28p^6 f_8 - 36p^7 f_9)\varrho. \end{aligned}$$

For a generic choice of the functions f_0, \dots, f_9 , the metric (3.11) has holonomy equal to \mathbf{G}_2 .

Next we present two more complicated examples. These will be examples of ambient metrics $\tilde{g}_{\mathcal{D}_{f,h}}$ with full \mathbf{G}_2 holonomy, which are also in closed form, but which are *not polynomial* in the variable p . We construct them by finding special closed form solutions to the second order system (3.6). The first of the examples gives a strategy how to find solutions of this type.

Example 3.2. Consider the distribution $\mathcal{D}_{f,h}$ with $f(x, y, p) = \sin(p)$ and $h(x, y) = 0$.

In order to find the ambient metric for $g_{\mathcal{D}_{f,h}}$, we first solve the equation $L(A) = f_{pppp}$ for $A(x, y, p, \varrho)$ and then determine B and C from the first order equations (3.7) and the initial conditions $B|_{\varrho=0} = 0$ and $C|_{\varrho=0} = 0$. To solve for A , we remark that if $f = f(x, y, p)$ satisfies a constant coefficient linear ODE with respect to the variable p and with fundamental solutions $f_1(x, y, p), \dots, f_N(x, y, p)$, then the analytic solutions of

$$L(A) = f_{pppp}$$

are of the form

$$A = a_1(\varrho)f_1(x, y, p) + a_2(\varrho)f_2(x, y, p) + \dots + a_N(\varrho)f_N(x, y, p)$$

where the $a_2(\varrho), \dots, a_N(\varrho)$ satisfy a system of linear second order ODEs.

For the problem at hand, this implies that $A = a_1(\varrho)\sin(p) + a_2(\varrho)\cos(p)$, where

$$2\varrho a_{1,\varrho\varrho} - 3a_{1,\varrho} + \frac{1}{8}a_1 = \frac{9}{40}, \quad \varrho a_{2,\varrho\varrho} - 3a_{2,\varrho} + \frac{1}{8}a_2 = 0 \quad \text{and} \quad a_1(0) = a_2(0) = 0,$$

The analytic solutions to these equations are

$$a_1(\varrho) = \frac{3}{20}\varrho \cos\left(\frac{\sqrt{\varrho}}{2}\right) - \frac{9}{10}\sqrt{\varrho} \sin\left(\frac{\sqrt{\varrho}}{2}\right) - \frac{9}{5}\left(\cos\left(\frac{\sqrt{\varrho}}{2}\right) - 1\right) \quad \text{and} \quad a_2(\varrho) = 0.$$

Then we find that

$$B = b(\varrho) \cos(p) \quad \text{and} \quad C = c(\varrho) \sin(p),$$

where

$$\begin{aligned} b(\varrho) &= -\frac{1}{120}\varrho^{\frac{3}{2}} \sin\left(\frac{\sqrt{\varrho}}{2}\right) - \frac{1}{10}\varrho \cos\left(\frac{\sqrt{\varrho}}{2}\right) + \frac{1}{2}\sqrt{\varrho} \sin\left(\frac{\sqrt{\varrho}}{2}\right) + \cos\left(\frac{\sqrt{\varrho}}{2}\right) - 1 \\ c(\varrho) &= \frac{1}{2160}\varrho^2 \cos\left(\frac{\sqrt{\varrho}}{2}\right) - \frac{1}{120}\varrho^{\frac{3}{2}} \sin\left(\frac{\sqrt{\varrho}}{2}\right) - \frac{13}{180}\varrho \cos\left(\frac{\sqrt{\varrho}}{2}\right) + \frac{1}{3}\sqrt{\varrho} \sin\left(\frac{\sqrt{\varrho}}{2}\right) + \frac{2}{3}\left(\cos\left(\frac{\sqrt{\varrho}}{2}\right) - 1\right). \end{aligned}$$

Note that these functions are analytic in ϱ . Moreover, by Theorem 3.5 the holonomy of the resulting ambient metric given by A , B and C is equal to \mathbf{G}_2 .

Example 3.3. Similarly, for

$$F = q^2 + e^p.$$

the solutions of equations (3.6) are of the form

$$A = A(\varrho, p) = a(\varrho)e^p, \quad B = B(\varrho, p) = b(\varrho)e^p, \quad C = C(\varrho, p) = c(\varrho)e^p$$

for functions a, b, c of ϱ given by

$$\begin{aligned} a(\varrho) &= \frac{3}{20}\varrho \cosh\left(\frac{\sqrt{\varrho}}{2}\right) - \frac{9}{10}\sqrt{\varrho} \sinh\left(\frac{\sqrt{\varrho}}{2}\right) - \frac{9}{5}(\cosh\left(\frac{\sqrt{\varrho}}{2}\right) - 1) \\ b(\varrho) &= -\frac{1}{120}\varrho^{\frac{3}{2}} \sinh\left(\frac{\sqrt{\varrho}}{2}\right) + \frac{1}{10}\varrho \cosh\left(\frac{\sqrt{\varrho}}{2}\right) - \frac{1}{2}\sqrt{\varrho} \sinh\left(\frac{\sqrt{\varrho}}{2}\right) + \cosh\left(\frac{\sqrt{\varrho}}{2}\right) - 1 \\ c(\varrho) &= \frac{1}{2160}\varrho^2 \cosh\left(\frac{\sqrt{\varrho}}{2}\right) - \frac{1}{120}\varrho^{\frac{3}{2}} \sinh\left(\frac{\sqrt{\varrho}}{2}\right) + \frac{13}{180}\varrho \cosh\left(\frac{\sqrt{\varrho}}{2}\right) - \frac{1}{3}\sqrt{\varrho} \sinh\left(\frac{\sqrt{\varrho}}{2}\right) + \frac{2}{3}(\cosh\left(\frac{\sqrt{\varrho}}{2}\right) - 1). \end{aligned}$$

Again, by Theorem 3.5 we conclude that the holonomy of the ambient metric is equal to \mathbf{G}_2 .

Finally we give an example that shows that even when the sufficient condition $A_\varrho \neq 0$ in Theorem 3.5 is not satisfied, we can obtain ambient metrics with holonomy equal to \mathbf{G}_2 .

Example 3.4. Consider the conformal class given by $f \equiv 0$. For such a conformal class the ambient metric in (3.5) has $A \equiv 0$, $B = -\frac{3}{20}\varrho h_y$, and $C = \frac{2}{45}\varrho h^2 - \frac{1}{15}\varrho(ph_y + h_x)$. When taking h as simple as $h(x, y) = y$ and differentiating the curvature, we get that the ambient holonomy has dimension 14 and hence is equal to \mathbf{G}_2 .

3.4. Solutions non-smooth in ϱ . To find *all* solutions to the linear system (3.6), i.e., also the ones that are not smooth in ϱ , in analogy to Theorem 2.2 we observe that the two independent solutions to $L(\varrho^k) = 0$ are ϱ^0 and $\varrho^{5/2}$. Thus, the most general solution to the system (3.6) can be obtained by the following series

$$\begin{aligned} A &= \sum_{k=1}^{\infty} a_k(x, y, p) \varrho^k + \varrho^{5/2} \sum_{k=0}^{\infty} \alpha_k(x, y, p) \varrho^k, \\ B &= \sum_{k=1}^{\infty} b_k(x, y, p) \varrho^k + \varrho^{5/2} \sum_{k=0}^{\infty} \beta_k(x, y, p) \varrho^k, \\ C &= \sum_{k=1}^{\infty} c_k(x, y, p) \varrho^k + \varrho^{5/2} \sum_{k=0}^{\infty} \gamma_k(x, y, p) \varrho^k. \end{aligned}$$

Theorem 3.6. *The general solution to the linear system (3.6) is given by*

$$\begin{aligned}
A &= \frac{3}{5} \sum_{k=1}^{\infty} \frac{(2k-1)(2k-3)}{2^{2k}(2k)!} \frac{\partial^{(2k+2)} f}{\partial p^{(2k+2)}} \varrho^k + 60 \varrho^{5/2} \sum_{k=0}^{\infty} \frac{(k+2)(k+1)}{2^{2k}(2k+5)!} \frac{\partial^{2k} \alpha_0}{\partial p^{2k}} \varrho^k, \\
B &= -\frac{3}{20} \varrho h_y - \frac{1}{15} \sum_{k=1}^{\infty} \frac{(2k-1)(2k-3)(2k-5)}{2^{2k}(2k)!} \frac{\partial^{(2k+1)} f}{\partial p^{(2k+1)}} \varrho^k \\
&\quad + \frac{20}{3} \varrho^{5/2} \sum_{k=0}^{\infty} (k+1)(k+2) 2^{2k}(2k+5)! \left(9 \frac{\partial^{2k} \beta_0}{\partial p^{2k}} - 2k \frac{\partial^{(2k-1)} \alpha_0}{\partial p^{(2k-1)}} \right) \varrho^k, \\
C &= \frac{2}{45} \varrho h^2 - \frac{1}{15} \varrho (p h_y + h_x) + \frac{2}{135} \sum_{k=1}^{\infty} \left(\frac{(k-3)(2k-1)(2k-3)(2k-5)}{2^{2k}(2k)!} \frac{\partial^{2k} f}{\partial p^{2k}} \right) \varrho^k \\
&\quad + \frac{20}{27} \varrho^{5/2} \sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{2^{2k}(2k+5)!} \left(81 \frac{\partial^{2k} \gamma_0}{\partial p^{2k}} - 36k \frac{\partial^{(2k-1)} \beta_0}{\partial p^{(2k-1)}} + 2k(2k-1) \frac{\partial^{(2k-2)} \alpha_0}{\partial p^{(2k-2)}} \right) \varrho^k.
\end{aligned}$$

Here α_0, β_0 and γ_0 , are arbitrary functions of the variables x, y and p .

Note that the analytic solutions are totally determined by the distribution, i.e. by the functions f and h . On the other hand, the non-smooth part of the solutions is determined only by the functions α_0, β_0 and γ_0 and hence does not depend on the distribution $\mathcal{D}_{f,h}$ at all.

Remark 3.4. In relation to Remark 3.1 we observe that the non-smooth solutions to the second order system (3.6) that are given in Theorem 3.6, when taking $f = h = 0$ and $\alpha_0 = \alpha_0(x, y)$, do *not* satisfy the first order system (3.7). Indeed, in this case we get $A \equiv 0$ and the first two equations of (3.7) become $B_p = 0$ and $B_\varrho = 0$, which do not hold when choosing $\frac{\partial^2 \beta_0}{\partial p^2} \neq 0$.

3.5. Remarks on the holonomy of the non-smooth solutions. The general solution to the linear system (3.6) given in Theorem 3.6 enables us, via (3.5), to write down explicit Ricci-flat metrics $\tilde{g}_{\mathcal{D}_{f,h}}$ that truncate at a prescribed order in the variable ϱ . Here we use some of these solutions to indicate certain holonomy issues related to the ambient metric construction. Note that solutions with nontrivial $\varrho^{\frac{5}{2}+k}$ terms are only defined for $\varrho \geq 0$ and that they are only twice differentiable at $\varrho = 0$. Thus, considering non-smooth in ϱ ambient metrics and trying to study the holonomy questions we would need a notion of a holonomy of a pseudo-Riemannian manifold with a boundary. We are unaware of such setting for the holonomy. Therefore in the following, considering our examples, we will only talk about the holonomy of $\tilde{g}_{\mathcal{D}_{f,h}}$ on a domain where $\varrho > 0$.

In the discussion below we concentrate on the non-smooth ambient metrics (3.5) for the *flat* distribution for which $f \equiv 0$ and $h \equiv 0$. In such case the solution to the linear system (3.6) is

$$\begin{aligned}
A &= \varrho^{5/2} \sum_{k=0}^{\infty} 60 \frac{(k+2)(k+1)}{2^{2k}(2k+5)!} \frac{\partial^{2k} \alpha_0}{\partial p^{2k}} \varrho^k, \\
B &= \varrho^{5/2} \sum_{k=0}^{\infty} \frac{20}{3} \frac{(k+1)(k+2)}{2^{2k}(2k+5)!} \left(9 \frac{\partial^{2k} \beta_0}{\partial p^{2k}} - 2k \frac{\partial^{(2k-1)} \alpha_0}{\partial p^{(2k-1)}} \right) \varrho^k, \\
C &= \varrho^{5/2} \sum_{k=0}^{\infty} \frac{20}{27} \frac{(k+1)(k+2)}{2^{2k}(2k+5)!} \left(81 \frac{\partial^{2k} \gamma_0}{\partial p^{2k}} - 36k \frac{\partial^{(2k-1)} \beta_0}{\partial p^{(2k-1)}} + 2k(2k-1) \frac{\partial^{(2k-2)} \alpha_0}{\partial p^{(2k-2)}} \right) \varrho^k,
\end{aligned}$$

and depends on three arbitrary functions $\alpha_0, \beta_0, \gamma_0$ of variables x, y and p . As an illustration we discuss holonomy properties of the corresponding ambient metrics for a very simple example, in which we make a particular choice of these 3 functions.

Example 3.5. Let c be a real constant and set

$$\begin{aligned}\alpha_0 &= \beta(x) + p\alpha(x), & \beta_0 &= \varphi_0(x) + p\varphi_1(x) + 252cp^2\alpha(x), \\ \gamma_0 &= \varphi_3(x) + p\varphi_4(x) + \frac{1}{81}p^2(2268\varphi_2(x) - \beta(x) + 18\varphi_1(x)).\end{aligned}$$

This gives the following solution to the linear system (3.6),

$$\begin{aligned}A &= (252p\alpha(x) + \beta(x))\varrho^{5/2}, \\ B &= (9c - 1)\alpha(x)\varrho^{7/2} + (\varphi_0(x) + p\varphi_1(x) + 252cp^2\alpha(x))\varrho^{5/2}, \\ C &= (\varphi_2(x) + (\frac{1}{9} - 4c)p\alpha(x))\varrho^{7/2} + (\varphi_3(x) + p\varphi_4(x) + \frac{1}{81}p^2(18\varphi_1(x) + 2268\varphi_2(x) - \beta(x)))\varrho^{5/2},\end{aligned}$$

corresponding to the following *family* of ambient metrics for the *flat* conformal structure

$$\begin{aligned}(3.12) \quad \tilde{g} &= 2dtd(\varrho t) + t^2(8(dp - qdx)^2 - 6(dz - 2qdp + q^2dx)dx - 12(dy - pdx)dq + \\ &+ (252p\alpha + \beta)\varrho^{5/2}(dy - pdx)^2 + \\ &+ 6((9c - 1)\alpha\varrho^{7/2} + (\varphi_0 + p\varphi_1 + 252cp^2\alpha)\varrho^{5/2})(dy - pdx)dx + \\ &+ 9((\varphi_2 + (\frac{1}{9} - 4c)p\alpha)\varrho^{7/2} + (\varphi_3 + p\varphi_4 + \frac{1}{81}p^2(18\varphi_1 + 2268\varphi_2 - \beta))\varrho^{5/2})dx^2).\end{aligned}$$

We see that the obtained family of Ricci-flat metrics \tilde{g} depends on *seven* arbitrary functions $\alpha = \alpha(x), \beta = \beta(x), \varphi_0 = \varphi_0(x), \varphi_1 = \varphi_1(x), \dots, \varphi_4 = \varphi_4(x)$ of the variable x , and a real constant c . The metric \tilde{g} is an ambient metric for the flat conformal structure represented by a flat metric

$$g = 8(dp - qdx)^2 - 6(dz - 2qdp + q^2dx)dx - 12(dy - pdx)dq.$$

The holonomy properties of the family \tilde{g} are quite interesting. Generically these metrics have *full* $\mathfrak{so}(4, 3)$ holonomy. We illustrate the behaviour with a few special cases:

(i) In order to verify the statement about generic holonomy, we put

$$\beta(x) \equiv \varphi_0(x) \equiv \varphi_1(x) \equiv \dots \equiv \varphi_4(x) \equiv 0, \quad \alpha(x) \equiv 1,$$

and c an arbitrary constant. Then our calculations show the following: The dimensions of the vector spaces spanned by the derivatives of the curvature at the point with $x = y = z = p = 0$ and $q = \varrho = t = 1$ up to order 4 are 6, 15, 18, 20, 21, respectively. Hence, even in this special situation, the holonomy algebra must be equal to $\mathfrak{so}(4, 3)$.

(ii) Interestingly, the metrics \tilde{g} include, as special cases, metrics with holonomy *equal to* \mathbf{G}_2 . For this we put

$$\beta(x) \equiv \varphi_0(x) \equiv \varphi_1(x) \equiv \dots \equiv \varphi_4(x) \equiv 0, \quad c = 0.$$

Here our calculations show the following: The dimensions of the vector spaces spanned by the derivatives of the curvature at the point with $x = y = z = p = 0$ and $q = \varrho = t = 1$ up to order 5 are 4, 10, 11, 12, 13, 14, respectively, and do not increase when differentiating again. Hence, the holonomy algebra is 14-dimensional, and one can check that it is equal to \mathfrak{g}_2 .

(iii) Another choice of the free functions in (3.12) shows that the considered family of Ricci flat metrics may still have different holonomy algebra. For this we set

$$\alpha(x) \equiv \beta(x) \equiv \varphi_0(x) \equiv \varphi_1(x) \equiv \varphi_2(x) \equiv \varphi_4(x) \equiv 0, \quad c = 1/9.$$

Here our Maple calculations show that the dimensions of the vector spaces spanned by the derivatives of the curvature up to order 2 are 4, 9, 10, respectively, and do not increase when differentiating again. In fact, the holonomy algebra is a semidirect product of a 7-dimensional radical and 3-dimensional semisimple Lie algebra. It is *not* a subalgebra of \mathfrak{g}_2 .

4. GENERIC 3-DISTRIBUTIONS IN DIMENSION 6 AND THEIR AMBIENT METRICS

4.1. Generic 3-distributions in dimension 6 and the associated conformal classes. In this section we construct explicit examples of ambient metrics for conformal structures with metrics of signature $(3, 3)$ which are naturally associated to *generic* rank 3 distributions in dimension 6. Here *generic* means that the distribution satisfies

$$[\mathcal{D}, \mathcal{D}] + \mathcal{D} = \text{TM}^6.$$

The associated conformal structures were introduced by Bryant in [4] as structures that encode local invariants of such distributions. We will call them *Bryant's conformal structures*.

Since our purpose is to find new examples of ambient metrics, we will not consider all generic rank 3 distributions in dimension 6, but only a special subclass. We consider an open set \mathcal{U} near the origin in \mathbb{R}^6 with coordinates $(x^1, x^2, x^3, y^1, y^2, y^3)$. We define a rank 3 distribution \mathcal{D}_f as the annihilator of three 1-forms

$$\theta_1 = dy^1 + x^2 dx^3, \quad \theta_2 = dy^2 + f dx^1, \quad \theta_3 = dy^3 + x^1 dx^2.$$

Here $f = f(x^1, x^2, x^3)$ is a differentiable function of the variables (x^1, x^2, x^3) only. Explicitly, the distribution is defined by

$$\mathcal{D}_f = \text{Span} \left(\frac{\partial}{\partial x^3} - x^2 \frac{\partial}{\partial y^1}, \frac{\partial}{\partial x^1} - f \frac{\partial}{\partial y^2}, \frac{\partial}{\partial x^2} - x^1 \frac{\partial}{\partial y^3} \right).$$

This distribution is generic provided that

$$\frac{\partial f}{\partial x^3} \neq 0,$$

and this will be always assumed in what follows. In the special case of a 3-distribution \mathcal{D}_f defined above, Bryant's conformal class $[g_{\mathcal{D}_f}]$ can be represented by a metric $g_{\mathcal{D}_f}$ as

$$(4.1) \quad g_{\mathcal{D}_f} = 2\theta^1\theta^4 + 2\theta^2\theta^5 + 2\theta^3\theta^6,$$

with the null co-frame

$$(4.2) \quad \begin{aligned} \theta^1 &= dy^1 + x^2 dx^3, & \theta^2 &= dy^2 + f dx^1, & \theta^3 &= dy^3 + x^1 dx^2, & \theta^4 &= \frac{3}{2}f_3^2 dx^1, \\ \theta^5 &= \frac{3}{2}f_3 dx^2 - \frac{1}{2}f_{33} \theta^1, \\ \theta^6 &= \frac{3}{2}f_3^2 dx^3 + \frac{1}{2}f_{13} \theta^2 + \frac{3}{2}f_3 f_2 dx^2 + \frac{1}{2}(f_3 f_{23} - f_2 f_{33}) \theta^1 + \frac{1}{2}(f_2 f_{13} - f_3 f_{12}) \theta^3. \end{aligned}$$

In these formulae we denote the partial derivatives with respect to the variables x^i by a subscript, for example, $f_1 = \frac{\partial f}{\partial x^1}$ or $f_{13} = \frac{\partial^2 f}{\partial x^3 \partial x^1}$. We should point out that, if we scale the first three forms in this co-frame by $\frac{1}{f_3}$ and the last three by $-\frac{3}{2}f_3$, then we obtain a co-frame which satisfies the structure equations (2.17) in [4].

We will now restrict to the distributions \mathcal{D}_f with $f = f(x^1, x^3)$, however all the following results will also be true for $f = f(x^2, x^3)$. The metrics in (4.1) defined by $f = f(x^1, x^3)$ have properties that are remarkably similar to those of pp-waves. To describe these, we denote by \mathcal{A} the algebra of functions depending only on x^1 and x^3 .

Lemma 4.1. *The metric $g_{\mathcal{D}_f}$ in (4.1) defined by $f \in \mathcal{A}$ has the following properties:*

- (i) *For $k \in \{2, 4, 6\}$ we have $\nabla \theta^k \in \text{span}_{\mathcal{A}}\{\theta^i \otimes \theta^j \mid i, j \in \{2, 4, 6\}\}$.*
- (ii) *The 3-form*

$$\kappa = (f_3)^{-1/3} \theta^2 \wedge \theta^4 \wedge \theta^6$$

is parallel, $\nabla \kappa = 0$.

(iii) The rank 3 distribution $\mathcal{V} := \{X \in T\mathcal{U} \mid X \lrcorner \kappa = 0\}$ is parallel, i.e.,

$$\nabla_X V \in \Gamma(\mathcal{V}), \quad \text{for all } V \in \Gamma(\mathcal{V}) \text{ and all } X \in T\mathcal{U}.$$

In particular, \mathcal{V} is integrable and, in fact, spanned by $\frac{\partial}{\partial x^2}$, $\frac{\partial}{\partial y^1}$ and $\frac{\partial}{\partial y^3}$.

Proof. To check the first point is a direct computation. For (ii), take a function $h = h(x^1, x^3)$, set $\phi = h \theta^2 \wedge \theta^4 \wedge \theta^6$, and compute

$$\nabla \phi = \frac{2}{9f_3^3} ((3h_3 f_3 + h f_{33})(2\theta^6 - f_{13}\theta^2) + 2(3h_1 f_3 + h f_{13})\theta^4) \otimes \theta^2 \wedge \theta^4 \wedge \theta^6.$$

Then $h = (f_3)^{-1/3}$ gives the required κ . Since κ is parallel, the distribution \mathcal{V} is parallel as well. \square

4.2. Bryant's conformal structures with vanishing obstruction and their ambient metrics. Recall that in even dimensions n , not every conformal class $[g]$ admits a smooth ambient metric. When proving their results in [8, 9], Fefferman and Graham fixed a metric g_0 in the conformal class and performed the *power series expansion* of $g(x^i, \varrho)$ in the variable ϱ ,

$$(4.3) \quad g(x^i, \varrho) = g_0 + 2\varrho P + \varrho^2 \mu + \dots$$

It follows that P is the Schouten tensor for g_0 and, if $n > 4$, that μ is given by

$$(4.4) \quad \mu_{ij} = \frac{1}{4-n} B_{ij} + P_i^k P_{kj}.$$

Here B is the *Bach tensor* of the metric g_0 . It is defined by

$$B_{ij} = \nabla^k C_{ijk} - P^{kl} W_{kijl},$$

where W_{ijkl} is the *Weyl tensor* and C_{ijk} the *Cotton tensor* of g_0 ,

$$C_{ijk} = \nabla_k P_{ij} - \nabla_j P_{ik}.$$

In dimension $n = 6$, for the power expansion of $g(x, \varrho)$ to continue beyond the ϱ^2 term, the metric g_0 representing the conformal class $[g_0]$ must satisfy

$$(4.5) \quad \begin{aligned} \mathcal{O}_{ij} = \nabla^k \nabla_k B_{ij} - 2W_{kijl} B^{kl} - 4P_k^k B_{ij} + 8P^{kl} \nabla_l C_{(ij)k} - 4C_i^k{}^l C_{ljk} + \\ + 2C_i^k{}^{kl} C_{jkl} + 4C_{ij}^l \nabla_l P_k^k - 4W_{kijl} P_m^k P^{ml} \equiv 0. \end{aligned}$$

In particular, when the *obstruction tensor* \mathcal{O}_{ij} vanishes, the metric $g(x, \varrho) = g_0 + 2P\varrho + \mu\varrho^2$, when inserted in the formula (1.1) for \tilde{g} , provides an example of an ambient metric

$$\tilde{g} = 2d(\varrho t)dt + t^2(g_0 + 2P\varrho + \mu\varrho^2),$$

that is, satisfying $\text{Ric}(\tilde{g}) = 0$.

We will now return to the distributions \mathcal{D}_f with $f = f(x^1, x^3)$, i.e. $f \in \mathcal{A}$, because for these the obstruction tensor for conformal classes of metrics in (4.1) vanishes. Again, all the following remains true for $f = f(x^2, x^3)$.

Proposition 4.1. *In the case when $f = f(x^1, x^3)$ the conformal classes $[g_{\mathcal{D}_f}]$ defined by the metrics $g_{\mathcal{D}_f}$ have the following properties:*

- (i) *Generically they are not conformally Cotton, and hence not conformally Einstein.*
- (ii) *They are Bach-flat and their Schouten tensor squares to zero, $P_i^k P_{kj} = 0$. In particular, the tensor μ in (4.4) of the quadratic terms in the expansion (4.3) vanishes.*
- (iii) *Each term in the formula (4.5) for the Fefferman-Graham obstruction tensor \mathcal{O} vanishes separately. Hence, $\mathcal{O} \equiv 0$.*
- (iv) *An ambient metric for all such conformal classes is given by*

$$\tilde{g}_f = 2d(\varrho t)dt + t^2(g_{\mathcal{D}_f} + 2\varrho P).$$

Proof. The fundamental observation is that the Schouten tensor P of $g_{\mathcal{D}_f}$ with $f \in \mathcal{A}$ satisfies⁵

$$(4.6) \quad P \in \text{span}_{\mathcal{A}}\{\theta^i \theta^j \mid i, j \in \{2, 4, 6\}\}$$

and

$$\nabla P \in \text{span}_{\mathcal{A}}\{\theta^i \otimes \theta^j \otimes \theta^k \mid i, j, k \in \{2, 4, 6\}\}.$$

Property (4.6) implies that the Schouten tensor is 2-step nilpotent, $P_{ik}P_j^k = 0$. Moreover, the Weyl tensor W satisfies

$$W(X, \cdot, \cdot, \cdot) \in \text{span}\{\theta^i \otimes \theta^j \otimes \theta^k \mid i, j, k \in \{2, 4, 6\}\}$$

whenever $X \in \mathcal{V}$. These properties for the Schouten and the Weyl tensor imply the vanishing of the Bach tensor and of all other terms in the formula (4.5) for \mathcal{O} .

That the metrics generically are not conformally Cotton-flat can be seen by checking that generically there is no vector Υ^i such that

$$C_{ijk} + \Upsilon^\ell W_{\ell ijk} = 0,$$

which is a necessary condition for a metric being conformal to a Cotton-flat metric [11].

The formula for an ambient metric then follows immediately from (4.3). It is a straightforward computation to check that \tilde{g}_f is indeed Ricci-flat. \square

Although Proposition 4.1 gives us an explicit formula for an ambient metric for Bryant's conformal classes $[g_{\mathcal{D}_f}]$ with $f = f(x^1, x^3)$ or $f = f(x^2, x^3)$, the ambient metrics are not unique. In order to find additional ambient metrics and exhibit the ambiguity at order 3 in ϱ , we again make an ansatz for these higher order terms to have the same form as Schouten tensor P of $g_{\mathcal{D}_f}$ as given in (4.6). Analogously to pp-waves and generic rank 2 distributions in dimension 5, our ansatz is

$$(4.7) \quad \tilde{g}_{f,S} = 2d(\varrho t)dt + t^2(g_{\mathcal{D}_f} + 2\varrho P + S(x^1, x^3, \varrho)),$$

with the tensor $S = S(x^1, x^3, \varrho)$ as

$$(4.8) \quad S(x^1, x^3, \varrho) = S_{22}(\theta^2)^2 + 2S_{24}\theta^2\theta^4 + 2S_{26}\theta^2\theta^6 + S_{44}(\theta^4)^2 + 2S_{46}\theta^4\theta^6 + S_{66}(\theta^6)^2.$$

We now impose the Ricci-flatness condition on $\tilde{g}_{f,S}$ and remarkably get a similar result as in Theorem 3.1, this time showing the ambiguity at the ϱ^3 -term. A straightforward computation of the Ricci-tensor of $\tilde{g}_{f,S}$ yields:

Theorem 4.1. *Let*

$$\tilde{g}_{f,S} = 2d(\varrho t)dt + t^2(g_{\mathcal{D}_f} + 2\varrho P + S(x^1, x^3, \varrho)),$$

be a metric with $f = f(x^1, x^3)$, $g_{\mathcal{D}_f}$ a metric as in (4.1)-(4.2) with Schouten tensor P as in (4.6), and with the tensor $S = S(x^1, x^3, \varrho)$ as in (4.8). Then $\tilde{g}_{f,S}$ is an ambient metric for the conformal class of Bryant's metrics $[g_{\mathcal{D}_f}]$ if and only if all the functions $S_{ij} = S_{ij}(x^1, x^3, \varrho)$ satisfy the same linear PDE,

$$\varrho \frac{\partial^2 S_{ij}}{\partial \varrho^2} - 2 \frac{\partial S_{ij}}{\partial \varrho} = 0.$$

The general solution S satisfying the initial conditions $S_{ij}(x^1, x^3, 0) = 0$, is given by $S = \varrho^3 Q$ with

$$Q(x^1, x^3) = Q_{22}(\theta^2)^2 + 2Q_{24}\theta^2\theta^4 + 2Q_{26}\theta^2\theta^6 + Q_{44}(\theta^4)^2 + 2Q_{46}\theta^4\theta^6 + Q_{66}(\theta^6)^2,$$

where the functions $Q_{ij}(x^1, x^3)$ are arbitrary. Every such Q gives an ambient metric for $g_{\mathcal{D}_f}$ by

$$(4.9) \quad \tilde{g}_{f,Q} = 2d(\varrho t)dt + t^2\left(g_{\mathcal{D}_f} + 2\varrho P + \varrho^3 Q(x^1, x^3)\right),$$

with P the Schouten tensor of $g_{\mathcal{D}_f}$ as in (4.6).

⁵The explicit formulae for the Schouten tensor components are not needed here and can be found in the Maple worksheet at http://digitalcommons.usu.edu/dg_applications/.

Remark 4.1. There is an analogous Theorem for $[g_{\mathcal{D}_f}]$ with $f = f(x^2, x^3)$.

The tensor Q , which is trace free with respect to $g_{\mathcal{D}_f}$, is responsible for the “Fefferman-Graham ambiguity”, i.e., for the ambiguity of ambient metrics for conformal classes in even dimensions. In general, it appears in the curvature tensor and thus cannot be gauged away by coordinate transformations.

4.3. On the holonomy of the ambient metrics. Now we will study the holonomy of ambient metrics for conformal classes $[g_{\mathcal{D}_f}]$ with $f \in \mathcal{A}$. For ambient metrics with $Q \neq 0$ we get a similar result as for pp-waves. For the ambient metrics with $Q = 0$, we show that their holonomy is contained in $\mathbf{Spin}(4, 3)$. Our terminology regarding invariance and stabilizers is explained in Remark 2.2.

Proposition 4.2. *Let $f = f(x^1, x^3)$ be a smooth function of two variables, $[g_{\mathcal{D}_f}]$ be Bryant’s conformal structure represented by a metric $g_{\mathcal{D}_f}$ as in (4.1), and let*

$$\tilde{g}_{f,Q} = 2d(\varrho t)dt + t^2(g_{\mathcal{D}_f} + 2\varrho P + \varrho^3 Q),$$

be one of its ambient metrics with $Q \in \text{span}_{\mathcal{A}}\{\theta^i \theta^j \mid i, j \in \{2, 4, 6\}\}$. Then:

- (i) *If $\kappa = (f_3)^{-1/3} \theta^2 \wedge \theta^4 \wedge \theta^6$ is the 3-form defined in Lemma 4.1, then the 4-form*

$$\tilde{\kappa} = -\frac{2t^3}{9} dt \wedge \kappa = -\frac{2t^3}{9(f_3)^{1/3}} dt \wedge \theta^2 \wedge \theta^4 \wedge \theta^6,$$

is parallel, i.e., $\tilde{\nabla} \tilde{\kappa} = 0$.

- (ii) *The totally null distribution $\tilde{\mathcal{V}}$ of tangent vectors with $X \lrcorner \tilde{\kappa} = 0$, which is spanned by $\frac{\partial}{\partial x^2}$, $\frac{\partial}{\partial y^1}$, $\frac{\partial}{\partial y^3}$ and ∂_{ϱ} , is parallel.*

- (iii) *The holonomy algebra of $\tilde{g}_{f,Q}$ is contained in the stabilizer of $\tilde{\kappa}$,*

$$\mathfrak{hol}(\tilde{g}_{f,Q}) \subset \mathfrak{sl}_4 \mathbb{R} \ltimes \Lambda_4 \mathbb{R} = \left\{ \begin{pmatrix} X & Z \\ 0 & -X^\top \end{pmatrix} \mid X \in \mathfrak{sl}_4 \mathbb{R}, Z + Z^\top = 0 \right\} \subset \mathfrak{so}(4, 4),$$

where $\Lambda_4 \mathbb{R}$ denotes the skew symmetric 4×4 matrices⁶. The representation of $\mathfrak{sl}_4 \mathbb{R}$ on the Abelian ideal $\Lambda_4 \mathbb{R}$ is given by $X \cdot Z = XZ - (XZ)^\top$.

Proof. The fact that κ is parallel for $g_{\mathcal{D}_f}$ suggests to consider the 4-form $dt \wedge \kappa$. One checks that

$$\tilde{\nabla}(dt \wedge \kappa) = -\frac{3}{t} dt \otimes (dt \wedge \kappa).$$

Hence, rescaling $dt \wedge \kappa$ to $t^3 dt \wedge \kappa$ gives a parallel form (the factor $-2/9$ will become clear in the next proposition). Since $\tilde{\kappa}$ is decomposable it defines a rank 4 distribution $\tilde{\mathcal{V}}$ that is parallel. In fact, $\tilde{\mathcal{V}} = \mathbb{R} \cdot \partial_{\varrho} \oplus \mathcal{V}$, where \mathcal{V} was defined in Lemma 4.1. Note that $\tilde{\mathcal{V}}$ is totally null. The parallel null distribution $\tilde{\mathcal{V}}$ is invariant under the holonomy and its stabilizer in $\mathfrak{so}(4, 4)$ is $\mathfrak{gl}_4 \mathbb{R} \ltimes \Lambda_4 \mathbb{R}$. That $\tilde{\kappa}$ is invariant under the holonomy algebra, reduces the holonomy further to $\mathfrak{sl}_4 \mathbb{R} \ltimes \Lambda_4 \mathbb{R}$. \square

Remark 4.2. Note that $\mathfrak{sl}_4 \mathbb{R} \simeq \mathfrak{so}(3, 3)$ as Lie algebra. Moreover, the representation of $\mathfrak{sl}_4 \mathbb{R}$ on the Abelian ideal $\Lambda_4 \mathbb{R}$ in the above $\mathfrak{sl}_4 \mathbb{R} \ltimes \Lambda_4 \mathbb{R}$ is just the representation of $\mathfrak{sl}_4 \mathbb{R}$ on $\Lambda^2 \mathbb{R}^4 \simeq \Lambda_4 \mathbb{R}$. The latter carries an $\mathfrak{sl}_4 \mathbb{R}$ -invariant bilinear form of signature $(3, 3)$ defined by the relation

$$\sigma \wedge \xi = \langle \sigma, \xi \rangle e_1 \wedge e_2 \wedge e_3 \wedge e_4.$$

Hence, we have

$$\mathfrak{hol}(\tilde{g}_{f,Q}) \subset \mathfrak{sl}_4 \mathbb{R} \ltimes \Lambda_4 \mathbb{R} \simeq \mathfrak{po}(3, 3),$$

where $\mathfrak{po}(3, 3)$ denotes the Poincaré algebra in signature $(3, 3)$, $\mathfrak{po}(3, 3) = \mathfrak{so}(3, 3) \ltimes \mathbb{R}^{3,3}$.

⁶We avoid the notation $\mathfrak{so}(4)$ here, because it would obscure the fact that $\Lambda_4 \mathbb{R}$ is an Abelian ideal in $\mathfrak{sl}_4 \mathbb{R} \ltimes \Lambda_4 \mathbb{R}$.

When writing $\mathfrak{hol}(\tilde{g}_{f,Q}) \subset \mathfrak{po}(3,3) \subset \mathfrak{so}(4,4)$ we refer to this representation and *not* to the more familiar representation of $\mathfrak{po}(3,3)$ in $\mathfrak{so}(4,4)$ as the stabilizer of a null vector. The same applies in the following, when we consider $\mathfrak{po}(3,2) = \mathfrak{so}(3,2) \ltimes \mathbb{R}^{3,2} \subset \mathfrak{po}(3,3)$.

Since $\mathfrak{po}(3,3)$ and $\mathfrak{spin}(4,3)$ have the same dimension, but are different, we obtain:

Corollary 4.1. *Ambient metrics (4.9) for Bryant's conformal structures with $f = f(x^1, x^3)$ or $f = f(x^2, x^3)$ cannot have holonomy equal to $\mathfrak{spin}(4,3)$.*

The next example shows that $\mathfrak{po}(3,3)$ is actually attained as holonomy group.

Example 4.1. We set $f = x^1(x^3)^2$ and $Q = \theta^2\theta^6$. In this case it turns out that the holonomy algebra of the ambient metric (4.9) is 21-dimensional. Hence, by Proposition 4.2 it is the Poincaré algebra in signature $(3,3)$,

$$\mathfrak{hol}_p(\tilde{g}_{f,Q}) \simeq \mathfrak{po}(3,3) = \mathfrak{so}(3,3) \ltimes \mathbb{R}^{3,3}.$$

Remark 4.3. Since the conformal holonomy is contained in the ambient holonomy, the result in [19, Theorem 1] implies that a conformal class $[g_{\mathcal{D}_f}]$ with $f = f(x^1, x^3)$ contains a certain preferred metric g_0 . This metric admits a parallel totally null rank 3 distribution which contains the image of the Ricci-tensor (or equivalently, of the Schouten tensor). We have already established in Lemma 4.1 and in the proof of Proposition 4.1 that the metric $g_{\mathcal{D}_f}$ is equal to this metric g_0 , which shows that we have chosen a suitable conformal factor.

Now we turn to ambient metrics with $Q = 0$. From now on we will assume that $f = f(x^1, x^3)$ is a smooth function of two variables, $[g_{\mathcal{D}_f}]$ Bryant's conformal structure represented by a metric $g_{\mathcal{D}_f} = 2\theta^1\theta^4 + 2\theta^2\theta^5 + 2\theta^3\theta^6$ as in (4.1), with Schouten tensor P , and that

$$\tilde{g}_f = 2d(\varrho t)dt + t^2(g_{\mathcal{D}_f} + 2\varrho P),$$

is the ambient metric with $Q = 0$. Then, a direct computation shows:

Lemma 4.2. *The 2-form*

$$\alpha = -\frac{9t}{2} f_3^{4/3} dt \wedge \theta^2 + \frac{t^2}{f_3^{5/3}} (\theta^4 \wedge \theta^6 + \frac{5}{2} f_{13} \theta^2 \wedge \theta^4 + 2 f_{33} \theta^2 \wedge \theta^6)$$

is parallel for \tilde{g}_f and squares to $\tilde{\kappa}$, i.e., $\alpha \wedge \alpha = \tilde{\kappa}$.

The key step in simplifying the calculations from now on is to introduce a co-frame ω^i in which the metric \tilde{g}_f , the parallel 2-form α , and the parallel 4-form $\tilde{\kappa}$ can be written as follows:

$$\begin{aligned} \tilde{g}_f &= \omega^1\omega^5 + \omega^2\omega^6 + \omega^3\omega^7 + \omega^4\omega^8, \\ \alpha &= \omega^5 \wedge \omega^8 + \omega^6 \wedge \omega^7, \\ \tilde{\kappa} &= \omega^{5678}, \quad \text{where we denote } \omega^{ijkh} = \omega^i \wedge \omega^j \wedge \omega^h \wedge \omega^k. \end{aligned}$$

This means that the parallel null distribution $\tilde{\mathcal{V}}$ is spanned by E_1, \dots, E_4 , where E_i is the dual frame to ω^i , $\omega^i(E_j) = \delta^i_j$. Such a co-frame is given by

$$\begin{aligned}\omega^1 &= \varrho dt + t d\varrho - \frac{t\varrho}{9f_3^3} (5f_{13}\theta^4 + 4f_{33}\theta^6), \\ \omega^2 &= t^2 \left(-\frac{5f_{13}}{9f_3^3} d\varrho + \theta^1 + \frac{\varrho(85f_{33}^2 - 27f_3f_{113})}{81f_3^6} \theta^4 \right), \\ \omega^3 &= -\frac{4f_{33}}{9f_3^{4/3}} d\varrho + f_3^{5/3} \theta^3 + \frac{2\varrho(80f_{33}f_{13} - 27f_3f_{133})}{81f_3^{13/3}} \theta^4 + \frac{\varrho(76f_{33}^2 - 27f_3f_{333})}{81f_3^{13/3}} \theta^6, \\ \omega^4 &= t \left(s_1 \theta^2 + s_2 \theta^4 - \frac{2}{9f_3^{4/3}} \theta^5 + s_3 \theta^6 \right), \\ \omega^5 &= dt + \frac{t}{9f_3^3} (5f_{13}\theta^4 + 4f_{33}\theta^6), \quad \omega^6 = \theta^4, \quad \omega^7 = \frac{t^2}{f_3^{5/3}} \theta^6, \quad \omega^8 = -\frac{9f_3^{4/3}}{2} \theta^2,\end{aligned}$$

where the functions s_1 , s_2 and s_3 are defined by

$$\begin{aligned}s_1 &:= \frac{\varrho(5f_3f_{13}^2f_{333} - 20f_{13}^2f_{33}^2 + 3f_3^2f_{133}^2 + 4f_3f_{33}^2f_{113} - 3f_3^2f_{113}f_{333})}{486f_3^{22/3}}, \\ s_2 &= -\frac{2\varrho(5f_3f_{13}f_{133} - 20f_{33}f_{13}^2 + 4f_3f_{33}f_{113})}{243f_3^{22/3}}, \\ s_3 &:= -\frac{2\varrho(5f_3f_{13}f_{333} - 20f_{13}f_{33}^2 + 4f_3f_{33}f_{133})}{243f_3^{22/3}}.\end{aligned}$$

Using this co-frame, the computations simplify and we are able to determine the parallel 4-forms and the holonomy of \tilde{g}_f when $Q = 0$. To this end, using the frame ω^i , we define the 4 form

$$(4.10) \quad \hat{\phi} = \omega^{1256} + \omega^{1357} - \omega^{1458} - 2\omega^{1467} - 2\omega^{2358} - \omega^{2367} + \omega^{2468} + \omega^{3478},$$

and the function

$$(4.11) \quad s = \frac{2\varrho(80f_{33}f_{13} - 27f_3f_{133})}{81f_3^{13/3}}.$$

Then we can show:

Proposition 4.3. *Let $f = f(x^1, x^3)$ a smooth function of two variables, $[g_{\mathcal{D}_f}]$ Bryant's conformal structure represented by a metric $g_{\mathcal{D}_f} = 2\theta^1\theta^4 + 2\theta^2\theta^5 + 2\theta^3\theta^6$ as in (4.1), and let*

$$\tilde{g}_f = 2d(\varrho t)dt + t^2(g_{\mathcal{D}_f} + 2\varrho P),$$

be the ambient metric with $Q = 0$. Then:

- (i) *If $\tilde{\kappa} = \omega^{5678}$ is the parallel 4-form of Lemma 4.2, $\hat{\phi}$ the 4-form in (4.10) and s the function in (4.11), then the 4-forms*

$$\begin{aligned}\beta &= \alpha \wedge (\omega^{15} + \omega^{26} + \omega^{37} + \omega^{48} - s\omega^{58}) = \omega^{1567} - \omega^{2568} - \omega^{3578} + \omega^{4678} - s\tilde{\kappa}, \quad \text{and} \\ \phi &= \hat{\phi} + s\beta + \frac{s^2}{2}\tilde{\kappa}\end{aligned}$$

are parallel, self-dual, and satisfy

$$\alpha \wedge \beta = 0, \quad \alpha \wedge \phi = -3 * \alpha = -3(\omega^{14} + \omega^{23}) \wedge \tilde{\kappa},$$

as well as

$$\beta \wedge \beta = \beta \wedge \phi = 0, \quad \phi \wedge \phi = 14 \omega^{12345678}.$$

- (ii) The holonomy algebra of \tilde{g}_f is contained in $\mathfrak{po}(3, 2)$, where $\mathfrak{po}(3, 2) = \mathfrak{so}(3, 2) \ltimes \mathbb{R}^{3,2}$ is the Poincaré algebra in signature $(3, 2)$ represented as explained in Remark 4.2.
- (iii) The holonomy algebra of \tilde{g}_f is contained in $\mathfrak{spin}(4, 3)$.

Proof. To check that β and ϕ are parallel is a direct computation. Note that, when $Q \neq 0$, in general they are not parallel.

Since the 2-form α of Lemma 4.2 is parallel, the holonomy algebra is contained in its stabilizer, i.e. in $\mathfrak{sp}_2(\mathbb{R}) \ltimes \Lambda_4\mathbb{R}$. Here $\mathfrak{sp}_2(\mathbb{R})$ is the symplectic Lie algebra in $\mathfrak{sl}_4(\mathbb{R})$ and again $\Lambda_4\mathbb{R}$ are the skew-symmetric 4×4 matrices. From α we obtain the skew adjoint endomorphism field \mathcal{J} by

$$\tilde{g}_f(\mathcal{J}X, Y) = \alpha(X, Y).$$

Since α is parallel, \mathcal{J} is parallel as well. In the fixed co-frame ω^i and its dual E_i , \mathcal{J} is of the form

$$\mathcal{J} = E_4 \otimes \omega^5 + E_3 \otimes \omega^6 - E_2 \otimes \omega^7 - E_1 \otimes \omega^8,$$

or, when written as a matrix with respect to E_i ,

$$\mathcal{J} = \begin{pmatrix} 0 & J \\ 0 & 0 \end{pmatrix}, \quad \text{with } J = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

First we notice that $\mathfrak{sp}_2\mathbb{R}$ acts trivially on J by its very definition. Secondly, since $\alpha \wedge \alpha \neq 0$, J is non-degenerate with respect to the signature $(3, 3)$ scalar product defined by the identification $\Lambda_4\mathbb{R} \simeq \Lambda^2\mathbb{R}^4$ (see Remark 4.2). Hence, $\Lambda_4\mathbb{R}$ splits invariantly under $\mathfrak{sp}_2\mathbb{R}$ as

$$\Lambda_4\mathbb{R} = (\mathbb{R} \cdot J)^\perp \oplus \mathbb{R} \cdot J = \mathbb{R}^{3,2} \oplus \mathbb{R}.$$

Recalling the isomorphism $\mathfrak{sp}_2\mathbb{R} \simeq \mathfrak{so}(3, 2)$, that is given by this representation, we obtain that the holonomy is contained in $(\mathfrak{so}(3, 2) \ltimes \mathbb{R}^{3,2}) \oplus \mathbb{R}$. Finally, with $\mathcal{J} \cdot \alpha = 0$, one can check that

$$\mathcal{J} \cdot \beta = \alpha \wedge \mathcal{J} \cdot (\omega^{15} + \omega^{26} + \omega^{37} + \omega^{48} - s\omega^{58}) = -2\alpha \wedge \alpha = -2\tilde{\kappa} \neq 0.$$

Since β is parallel and hence holonomy invariant, this implies that \mathcal{J} is not contained in the holonomy algebra, which is then further reduced to $\mathfrak{po}(3, 2) = \mathfrak{so}(3, 2) \ltimes \mathbb{R}^{3,2}$.

In order to check that the holonomy algebra is contained in $\mathfrak{spin}(4, 3)$ we introduce the co-frame

$$\begin{aligned} \xi^1 &= -\frac{a}{64} \omega^1 + a \omega^5, & \xi^2 &= \frac{a}{16} \omega^2 - \frac{a}{4} \omega^6, & \xi^3 &= \frac{a}{8} \omega^3 - \frac{a}{8} \omega^7, \\ \xi^4 &= \frac{a}{2} \omega^4 - \frac{a}{32} \omega^8, & \xi^5 &= \frac{a}{16} \omega^2 + \frac{a}{4} \omega^6, & \xi^6 &= -\frac{a}{64} \omega^1 - a \omega^5, \\ \xi^7 &= -\frac{a}{2} \omega^4 - \frac{a}{32} \omega^8, & \xi^8 &= -\frac{a}{8} \omega^3 - \frac{a}{8} \omega^7, & & \text{with } a = \sqrt{32}. \end{aligned}$$

This transforms the metric \tilde{g}_f and the 4-form $\hat{\phi}$ into

$$\begin{aligned} \tilde{g}_f &= -(\xi^1)^2 - (\xi^2)^2 - (\xi^3)^2 - (\xi^4)^2 + (\xi^5)^2 + (\xi^6)^2 + (\xi^7)^2 + (\xi^8)^2, \quad \text{and} \\ \hat{\phi} &= \xi^{1234} - \xi^{1256} + \xi^{1278} - \xi^{1357} - \xi^{1368} - \xi^{1458} + \xi^{1467} \\ &\quad + \xi^{2358} - \xi^{2367} - \xi^{2457} - \xi^{2468} + \xi^{3456} - \xi^{3478} + \xi^{5678}. \end{aligned}$$

Thus $\hat{\phi}$ is precisely the 4-form whose stabilizer defines the representation $\mathfrak{spin}(4, 3) \subset \mathfrak{so}(4, 4)$, where we use the conventions in [3]. Along $\varrho = 0$ the parallel form ϕ is equal to $\hat{\phi}$. This is sufficient to conclude that the stabilizer of ϕ is $\mathfrak{spin}(4, 3)$, which yields $\mathfrak{hol}(\tilde{g}_f) \subset \mathfrak{spin}(4, 3)$. \square

The next example shows that in general the holonomy does not reduce further than $\mathfrak{po}(3, 2)$, proving the inclusion $\mathfrak{po}(3, 2) \subset \mathfrak{spin}(4, 3)$.

Example 4.2. We set $f = x^1(x^3)^2$ and $Q = 0$. In this case we compute the holonomy of \tilde{g}_f to be a 15-dimensional Lie algebra $\mathfrak{hol}_p(\tilde{g}_f)$. Hence it is equal to $\mathfrak{po}(3, 2)$. Since we also have $\mathfrak{hol}_p(\tilde{g}_f) \subset \mathfrak{spin}(4, 3)$, this shows the inclusion $\mathfrak{po}(3, 2) \subset \mathfrak{spin}(4, 3)$.

Note that the null 4-plane \tilde{V} of vectors X such that $X \lrcorner \alpha = 0$ is the only subspace that is invariant under the holonomy. In particular, it does not admit any invariant lines, reflecting the fact that the conformal classes defined by $f = f(x^1, x^3)$ generically are not conformally Einstein.

Having the inclusion $\mathfrak{po}(3, 2) \subset \mathfrak{spin}(4, 3)$, we can summarize:

Theorem 4.2. *Let $f = f(x^1, x^3)$ be a smooth function of two variables, $[g_{\mathcal{D}_f}]$ be Bryant's conformal structure represented by a metric $g_{\mathcal{D}_f}$ as in (4.1) and with Schouten tensor P , and let*

$$\tilde{g}_f = 2d(\varrho t)dt + t^2(g_{\mathcal{D}_f} + 2\varrho P),$$

be the ambient metric with $Q = 0$. Then the holonomy of \tilde{g}_f is contained in $\mathfrak{po}(3, 2)$, and hence contained in $\mathfrak{spin}(4, 3)$.

Our last example is an ambient metric with the 7-dimensional Heisenberg algebra as holonomy.

Example 4.3. Now we set $f = x^1x^3$ and $Q = 0$. In this case it turns out that the holonomy algebra of the corresponding ambient metric is a 7-dimensional, 2-step nilpotent Lie algebra with a 1-dimensional centre. Hence, as a Lie algebra, $\mathfrak{hol}_p(\tilde{g}_f)$ is isomorphic to the 7-dimensional Heisenberg algebra $\mathfrak{he}_3(\mathbb{R})$ defined as

$$\mathfrak{he}_3(\mathbb{R}) = \left\{ \begin{pmatrix} 0 & u & r \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix} \middle| u, v \in \mathbb{R}^3, r \in \mathbb{R} \right\}.$$

In addition to the parallel forms described in Proposition 4.3, this ambient metric admits two parallel 1-forms, namely

$$\varphi^1 = (x^1)^{c_+} dt + \frac{c_+ t}{9(x^1)^{c_-}} \theta^4 \quad \text{and} \quad \varphi^2 = (x^1)^{c_-} dt + \frac{c_- t}{9(x^1)^{c_+}} \theta^4, \quad \text{with } c_{\pm} = \frac{1}{6}(9 \pm \sqrt{21}).$$

The parallel 1-forms φ^1 and φ^2 are linearly independent on a dense open set, null and orthogonal to each other. Since the conformal holonomy is contained in the ambient holonomy [2], φ^1 and φ^2 define two conformal standard tractors that are null and parallel for the normal conformal tractor connection. It is well-known (see for example [14, 18]) that parallel tractors, on a dense open set, correspond to local Einstein scales (Ricci-flat scales in case of null tractors). We conclude that the conformal class $[g_{\mathcal{D}_f}]$ defined by $f = x^1x^3$ contains two local Ricci-flat metrics. Indeed, one verifies that the metrics

$$(c_1(x^1)^{c_+} - c_2(x^1)^{c_-})^{-2} g_{\mathcal{D}_f},$$

in which c_1 and c_2 are constants, are Ricci-flat.

Remark 4.4 (Non-vanishing obstruction). We close the paper with the observation that *not all* of Bryant's conformal structures have vanishing obstruction tensor. For functions f that are more general than $f = f(x^1, x^3)$ or $f = f(x^2, x^3)$, the Fefferman-Graham obstruction does *not* vanish. This happens for example for the generic rank 3 distribution associated with the function

$$f = x^3 + x^1x^2 + (x^2)^2 + (x^3)^2.$$

For this f the obstruction tensor \mathcal{O}_{ij} in (4.5) is *not* zero — it has 13 non-vanishing components. Hence, the conformal class associated to this f does not admit an analytic ambient metric.

REFERENCES

- [1] I. M. Anderson, T. Leistner, and P. Nurowski. Conformal classes with linear Fefferman-Graham equations. In progress.
- [2] S. Armstrong and T. Leistner. Ambient connections realising conformal tractor holonomy. *Monatsh. Math.*, 152(4):265–282, 2007.
- [3] H. Baum and I. Kath. Parallel spinors and holonomy groups on pseudo-Riemannian spin manifolds. *Ann. Global Anal. Geom.*, 17(1):1–17, 1999.
- [4] R. L. Bryant. Conformal geometry and 3-plane fields on 6-manifolds. In *Developments of Cartan Geometry and Related Mathematical Problems*, volume 1502 of *RIMS Symposium Proceedings*, pages 1–15, 2006.
- [5] E. Cartan. Les systèmes de Pfaff, à cinq variables et les équations aux dérivées partielles du second ordre. *Ann. Sci. École Norm. Sup. (3)*, 27:109–192, 1910.
- [6] E. A. Coddington and R. Carlson. *Linear ordinary differential equations*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1997.
- [7] B. Doubrov and A. Govorov. A new example of a generic 2-distribution on a 5-manifold with large symmetry algebra. *Preprint, arXiv:1305.7297*, May 2013.
- [8] C. Fefferman and C. R. Graham. Conformal invariants. *Astérisque*, (Numero Hors Serie):95–116, 1985. The mathematical heritage of Élie Cartan (Lyon, 1984).
- [9] C. Fefferman and C. R. Graham. *The ambient metric*, volume 178 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2012.
- [10] A. R. Gover and F. Leitner. A sub-product construction of Poincaré-Einstein metrics. *Internat. J. Math.*, 20(10):1263–1287, 2009.
- [11] A. R. Gover and P. Nurowski. Obstructions to conformally Einstein metrics in n dimensions. *J. Geom. Phys.*, 56(3):450–484, 2006.
- [12] C. R. Graham and T. Willse. Parallel tractor extension and ambient metrics of holonomy split G_2 . *J. Differential Geom.*, 92(3):463–505, 2012.
- [13] I. Kath. $G_{2(2)}^*$ -structures on pseudo-Riemannian manifolds. *J. Geom. Phys.*, 27(3-4):155–177, 1998.
- [14] T. Leistner. Conformal holonomy of C-spaces, Ricci-flat, and Lorentzian manifolds. *Differential Geom. Appl.*, 24(5):458–478, 2006.
- [15] T. Leistner and P. Nurowski. Ambient metrics for n -dimensional pp -waves. *Comm. Math. Phys.*, 296(3):881–898, 2010.
- [16] T. Leistner and P. Nurowski. Ambient metrics with exceptional holonomy. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, XI:407–436, 2012.
- [17] T. Leistner and P. Nurowski. Conformal pure radiation with parallel rays. *Classical Quantum Gravity*, 29(5):055007, 15, 2012.
- [18] F. Leitner. Conformal Killing forms with normalisation condition. *Rend. Circ. Mat. Palermo (2) Suppl.*, (75):279–292, 2005.
- [19] A. Lischewski. Reducible conformal holonomy in any metric signature and application to twistor spinors in low dimension. Preprint, arXiv:1408.1685, Aug. 2014.
- [20] P. Nurowski. Differential equations and conformal structures. *J. Geom. Phys.*, 55(1):19–49, 2005.
- [21] A. Trautman. Clifford algebras and their representation. In J.-P. Francoise, G. L. Naber, and T. S. Tsun, editors, *Encyclopedia of Mathematical Physics*, volume 1, pages 518–530. Academic Press/Elsevier Science, Oxford, 2006.
- [22] T. Willse. An explicit ambient metric of holonomy G_2^* . Preprint, arXiv:1411.7172, Nov. 2014.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UTAH STATE UNIVERSITY, LOGAN UTAH, 84322, USA
E-mail address: Ian.Anderson@usu.edu

SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF ADELAIDE, SA 5005, AUSTRALIA
E-mail address: thomas.leistner@adelaide.edu.au

CENTRUM FIZYKI TEORETYCZNEJ PAN AL. LOTNIKÓW 32/46 02-668 WARSZAWA, POLAND
E-mail address: nurowski@cft.edu.pl